Preparing a Tridiagonal Skew-Hermitian Differentiation Matrix on the Real Line

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Shopping List

Find a basis $\{\varphi_n\}_{n=0}^{\infty} = \Phi \subset \mathrm{C}^{\infty}(\mathbb{R})$ for which $\mathscr{D} := {}_{\Phi}[\frac{\mathrm{d}}{\mathrm{d}x}]_{\Phi}$ is an infinite complex matrix such that

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- tridiagonal
  fast computation: computation to nth term is $\mathcal{O}(n)$
- skew-Hermitian
  stability: A skew-Hermitian $\Rightarrow \mathrm{e}^{A}$ unitary

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We call such a basis Φ, a *T-system*.

OPS Basics: What are They?

A sequence $\{p_n\}_{n=0}^{\infty} = P$ is called an *orthogonal polynomial system* with respect to weight $w : \mathbb{R} \to [0, \infty)$ if

- p_n is a polynomial of degree n,
- $\int p_m \overline{p_n} w = 0$ if and only if $m \neq n$.

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It is a straightforward exercise that monic OPS's follow the three term recurrance relation

$$\begin{cases} p_1(x) = (x - c_0)p_0(x), \\ p_n(x) = (x - c_n)p_{n-1}(x) - \lambda_n p_{n-2}(x) \qquad n = 2, 3, \dots, \end{cases}$$

where c_n, λ_n are real and $\lambda_n > 0$.

OPS Basics: The Jewel

In 1935 Jean Favard showed the rermarkable fact that the converse holds:
given real sequences $\{c_n\}, \{\lambda_n\}$ with $\lambda_n > 0$, there exist $\{p_n\}$ such that

$$\begin{cases} p_1(x) = (x - c_0)p_0(x), \\ p_n(x) = (x - c_n)p_{n-1}(x) - \lambda_n p_{n-2}(x), \end{cases}$$

for $n \geq 2$, and there exists a 'weight' w such that P is an OPS with respect to it.

Figure: Jean Favard, August 1963

OPS Basics: A Whole Zoo

Given positive-definite weight function $w(x) \geq 0$, one can use the Gram-Schmidt process to generate an ortho*normal* polynomial sequence from $\{1, x, x^2, \ldots\}$ with respect to inner product $\langle p, q \rangle_w := \int p\overline{q}w$.

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$w(x) = \mathrm{e}^{-x^2} \to$ Hermite polynomials H_n.

$w_\alpha(x) = \chi_{[0,\infty)}(x)x^\alpha \mathrm{e}^{-x} \to$ Laguerre polynomials $\mathrm{L}_n^{(\alpha)}$.

$w(x) = (1 - x^2)^{-1/2} \to$ Chebyshev poly. of the first kind T_n.

$w_{(\alpha,\beta)}(x) = (1 - x)^\alpha (1 + x)^\beta \to$ Jacobi polynomials $\mathrm{P}_n^{(\alpha,\beta)}$.

OPS Basics: A Whole Zoo Cont.

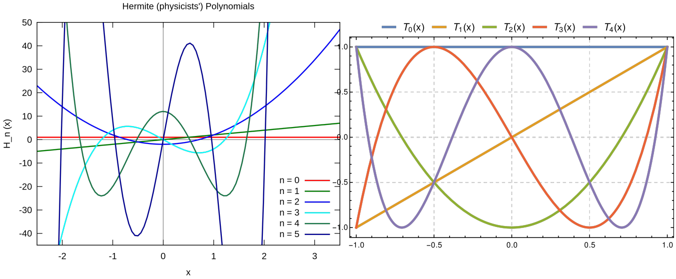

Figure: wikipedia commons

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Putting Pen to Paper

Multiplying by complex phase factors in $\{z \in \mathbb{C} : |z| = 1\}$, the criterion for $\mathscr{D}$ is WLOG equivalent to finding

$$\varphi_n' = \begin{cases} ic_0\varphi_0 + b_0\varphi_1 & n = 0 \\ -b_{n-1}\varphi_{n-1} + ic_n\varphi_n + b_n\varphi_{n+1} & n = 1, 2, \ldots \end{cases}$$

where b_n, c_n real and $b_n > 0$. Additionally, if φ_0 is even, then $c_n \equiv 0$.

A Curious Coincidence

Want:

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Have (Favard's theorem):

$$xp_n(x) = \begin{cases} c_0 p_0(x) + b_n p_1(x) & n = 0 \\ a_n p_{n-1}(x) + c_n p_n(x) + b_n p_{n+1}(x) & n = 1, 2, \ldots \end{cases}$$

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Where Φ is linearly independent.
Have (Favard's theorem):

$$xp_n(x) = \begin{cases} c_0p_0(x) + b_np_1(x) & n = 0 \\ a_np_{n-1}(x) + c_np_n(x) + b_np_{n+1}(x) & n = 1, 2, \ldots \end{cases}$$

Note: $\mathcal{F}\{f'\}(\xi) = i\xi\mathcal{F}\{f\}(\xi)$ where $\mathcal{F}\{f\}(\xi) := \int f(x)\mathrm{e}^{-ix\xi}\mathrm{d}x$.

The Trick

$$\varphi_n' = -b_{n-1}\varphi_{n-1} + ic_n\varphi_n + b_n\varphi_{n+1} \qquad (b_{-1} := 0)$$

$$\Leftrightarrow ix\widehat{\varphi_n} = -b_{n-1}\widehat{\varphi_{n-1}} + ic_n\widehat{\varphi_n} + b_n\widehat{\varphi_{n+1}} \qquad (\text{apply } \mathcal{F})$$

$$\Leftrightarrow xi^{-n}\widehat{\varphi_n} = b_{n-1}i^{-(n-1)}\widehat{\varphi_{n-1}} + c_n i^{-n}\widehat{\varphi_n} - b_n i^{-(n+1)}\widehat{\varphi_{n+1}}.$$

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Also want orthogonality; Parseval's theorem says $\int \widehat{f}(x)\overline{\widehat{g}(x)}\mathrm{d}x \equiv \int f(x)\overline{g(x)}\mathrm{d}x$, so (for orthogonality) it suffices to find P such that $\widehat{i^{-m}\varphi_m}\overline{\widehat{i^{-n}\varphi_n}} = p_m\overline{p_n}w$.

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Note $\varphi_n := i^n\mathcal{F}^{-1}\{p_n\sqrt{w}\}$ works. Plugging back in we get

$$xp_n(x) = b_{n-1}p_{n-1}(x) + c_n p_n(x) - b_n p_{n+1}(x)$$

So by Favard's theorem such a P always exists.

Basis of What?

$$\mathrm{Span}(\Phi) = \mathcal{F}\{\mathrm{Span} P\sqrt{w}\}$$

If $P\restriction_\Omega$ is dense in $\mathrm{L}_2(\Omega)$ where $\Omega := \mathrm{supp}(w)$, then

$$\mathrm{Span}(\Phi) = \{f \in \mathrm{L}_2(\mathbb{R}) : \mathcal{F}\{f\} \text{ has support on } \Omega\} =: \mathrm{PW}_\Omega(\mathbb{R})$$

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Find P with $\Omega = \mathbb{R}$ dense in $\mathrm{L}_2(\mathbb{R})$ $\Rightarrow$ Generate Φ with $\varphi_n := i^n \mathcal{F}^{-1}\{p_n\sqrt{w}\}$ $\Rightarrow$ Then Φ is a T-system for $\mathrm{PW}_\mathbb{R}(\mathbb{R}) = \mathrm{L}_2(\mathbb{R})$

Examples

Hermite ($w(x) = \mathrm{e}^{-x^2}$).
$\rightarrow$ Hermite functions $\varphi_n(x) = \mathrm{H}_n(x)\mathrm{e}^{-x^2}$, dense in $\mathrm{L}_2(\mathbb{R})$.

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Laguerre ($w(x) = \chi_{[0,\infty)}(x)\mathrm{e}^{-x}$).
$\to$ Malmquist-Takenaka functions $\varphi_n(x) = i^n\sqrt{\frac{2}{\pi}}\left(\frac{1+2ix}{1-2ix}\right)^n \frac{1}{1-2ix}$, dense in $\mathrm{PW}_{[0,\infty)}(\mathbb{R})$.

So $\{\varphi_n\}_{n=-\infty}^{\infty}$ gives 'T-system' dense in $\mathrm{PW}_{[0,\infty)}(\mathbb{R}) \oplus \mathrm{PW}_{(-\infty,0]}(\mathbb{R}) = \mathrm{L}_2(\mathbb{R})$.

$\tilde{w}(x) := w(-x)$ gives $\varphi_n(x) = i^{-n}\sqrt{\frac{2}{\pi}}\left(\frac{1+2ix}{1-2ix}\right)^{-n} \frac{1}{1-2ix}$, dense in $\mathrm{PW}_{(-\infty,0]}(\mathbb{R})$.

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Continuous Hahn ($w_{(a,b)}(x) = \frac{1}{2\pi}\Gamma(a+ix)\Gamma(b-ix)$).
$\to \varphi_n^{(a,b)}(x) = (1-\tanh x)^a(1+\tanh x)^b \mathrm{P}_n^{(2a-1,2b-1)}(\tanh x)$, dense in $\mathrm{L}_2(\mathbb{R})$.

Malmquist-Takenaka Functions Plot

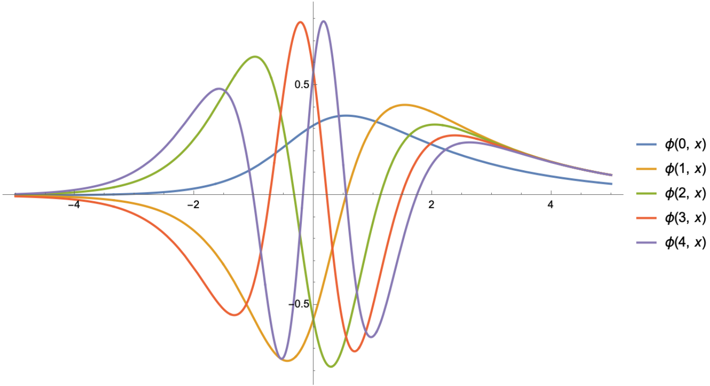

Figure: Malmquist-Takenaka Functions

Transformed Hahn Polynomials Plot

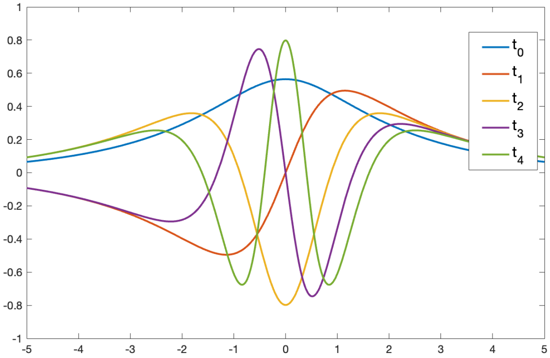

Figure: Transformed Hahn Polynomials

A Method on Fast Computation

Transformed Laguerre

$$\int_{-\infty}^{\infty} f(x)\overline{\varphi_n(x)}\mathrm{d}x = \frac{1}{\sqrt{2\pi}i^n}\int_{-\pi}^{\pi} f\left(\frac{1}{2}\tan\frac{\theta}{2}\right)\left(1 - i\tan\frac{\theta}{2}\right)\mathrm{e}^{-in\theta}\mathrm{d}\theta$$

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Transformed (Normalised) Continuous Hahn ($a = b = 1/4$; $a = b = 3/4$)

$$\int_{-\infty}^{\infty} f(x)\tau_n(x)\mathrm{d}x = \sqrt{\frac{1}{\pi}}\int_{0}^{\pi} f\left(\log\cot\frac{\theta}{2}\right)\frac{\cos n\theta}{\sqrt{\sin\theta}}\mathrm{d}\theta \qquad n = 0,$$

$$\int_{-\infty}^{\infty} f(x)\tau_n(x)\mathrm{d}x = \sqrt{\frac{2}{\pi}}\int_{0}^{\pi} f\left(\log\cot\frac{\theta}{2}\right)\frac{\cos n\theta}{\sqrt{\sin\theta}}\mathrm{d}\theta \qquad n \geq 1;$$

$$\int_{-\infty}^{\infty} f(x)\upsilon_n(x)\mathrm{d}x = \sqrt{\frac{2}{\pi}}\int_{0}^{\pi} f\left(\log\cot\frac{\theta}{2}\right)\frac{\sin(n+1)\theta}{\sqrt{\sin\theta}}\mathrm{d}\theta \qquad n \geq 0.$$

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