Preparing a Tridiagonal Skew-Hermitian Differentiation Matrix on the Real Line

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1 Introduction

The motivation for our work is incredibly simple: we investigate bases $\Phi = \{\varphi_n\}_{n=0}^{\infty} \subset \mathrm{C}^{\infty}(\mathbb{R})$ of the Hilbert space $\mathrm{L}_2(\mathbb{R})$ for which the differentiation matrix with respect to this basis $\mathscr{D} \in \mathrm{Mat}\infty(\mathbb{C})$ is skew-Hermitian and tridiagonal. The theoretical basis for most of the results here were first presented in [IW21].

Definition 1.1. We say $\Phi \subset \mathrm{C}^{\infty}(\mathbb{R})$ is a *T-system* if it is a linearly independent set for which $\mathscr{D}$ is skew-Hermitian and tridiagonal.

Through simple algebra it can be shown that

Proposition 1.1. *Suppose $\tilde{\Phi}$ is a T-system, then there exists a rescalled T-system $\Phi := \{\mathrm{e}^{i\theta_n}\tilde{\varphi_n} : \theta_n \in \mathbb{R}, \tilde{\varphi}_n \in \tilde{\Phi}\}$ with real coefficients b_n, c_n with $n = 0, 1, 2, \ldots$ such that*

$$\varphi_n' = \begin{cases} ic_0\varphi_0 + b_1\varphi_1 & n = 0 \\ -b_{n-1}\varphi_{n-1} + ic_n\varphi_n + b_n\varphi_{n+1} & n = 1, 2, \ldots \end{cases} \tag{1}$$

where $b_n > 0$. Moreover if φ_0 is even, then $c_n \equiv 0$.

Therefore without loss of generality we may assume any such T-system takes the form of eq. 1.

Now it is clear that the main appeal of such a basis would be the fast and stable computation of the derivative of a function written in such a basis: tridiagonality gives fast computation; skew-Hermitionality gives stability (note $\mathrm{e}^{\mathscr{D}}$ is unitary if the former holds). But how should one go about coming up with such a system in the first place? Well it turns out there's a clever trick one can do to produce infinitely many T-systems, in a manner intimitelay related to the theory of orthogonal polynomial sequences. We recall a classical theorem due to Favard.

Theorem 1.1 (Favard 1935)**.** *Let $P = \{p_n\}_{n=0}^{\infty}$ be a sequence of polynomials with $\deg p_n = n$. There exists a sequence of numbers b_n, c_n such that*

$$xp_n(x) = \begin{cases} c_0p_0(x) + b_1p_1(x) & n = 0 \\ -b_{n-1}p_{n-1}(x) + c_np_n(x) + b_np_{n+1}(x) & n = 1, 2, \ldots \end{cases} \tag{2}$$

where $b_n > 0$ if and only if there exists a Borel measure μ such that P is orthogonal with respect to the inner product $\langle p, q\rangle := \int_{\mathbb{R}} p\overline{q}\mathrm{d}\mu$.

The reader may through squinting their eyes notice a striking similarity between eq. 1 and eq. 2: this is delibrate. In fact the only real difference is that the left handside of both equations differ by the application of the derivative, and multiplication by the argument. Elementary familiarity with the Fourier transform allows us to establish a concrete connection.

Proposition 1.2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be sufficiently regular, then* $\mathcal{F}\{f'\}(\xi) = i\xi\mathcal{F}\{f\}(\xi)$ *where* $\mathcal{F}\{f\} : \xi \mapsto \int_{\mathbb{R}} f(x)\mathrm{e}^{-ix\xi}\mathrm{d}x$ *is the Fourier transform.*

Proof. Integration by parts. □

Naïvely Fourier transforming eq. 2 doesn't work as, firstly (and perhaps most importantly), the polynomials p_n are unbounded, and secondly, we get several unpleasent factors of i's. The trick is to define

$$\varphi_n := i^n\mathcal{F}^{-1}\{p_n\sqrt{w}\} \tag{3}$$

where $w \geq 0$ is the weight function corresponding to the borel measure μ. In effect we are multiplying eq. 2 by $\sqrt{w}$ and *then* Fourier transforiming, modulo factors of i. This way we recover eq. 1, and as a bonus we get that Φ is *orthonormal* in the $\mathrm{L}_2(\mathbb{R})$ norm[1] by Plancherel's theorem, so Φ is *a fortiori* linearly independent.

We summarise this section's main result

Proposition 1.3. *Let* P *be an orthonormal polynomial sequence with respect to weight* w *with support* Ω*. Let* Φ *be given by eq. 3, then* $\mathscr{D}$ *is tridiagonal and skew-Hermitian, and* Φ *is an orthonormal basis of* $\mathcal{F}\{\mathrm{Span}P\sqrt{w}\}$*. In particular if* $\mathrm{Span}P \restriction_\Omega$ *is dense in* $\mathrm{L}_2(\Omega)$*, then* $\mathrm{Span}\Phi = \{f \in \mathrm{L}_2(\mathbb{R}) : \mathrm{Supp}(\mathcal{F}\{f\}) \subset \Omega\} =: \mathrm{PW}_\Omega(\mathbb{R})$*, a Paley-Wiener space.*

Corollary 1.1. *Let* $\mathrm{Span}P$ *be dense in* $\mathrm{L}_2(\mathbb{R})$ *with* w *supported on the entire real line, then* Φ *is a T-system basis of* $\mathrm{L}_2(\mathbb{R})$*.*

2 Examples of T-systems derived from orthogonal polynomial sequences

2.1 Hermite polynomials

The Gaussian weight $w(x) = \mathrm{e}^{-x^2}$ gives us the Hermite polynomials. It turns out that with eq. 3 that the φ_n's are none other than the Hermite *functions*, which are proportional to the Hermite polynomials multiplied by $\mathrm{e}^{-x^2/2}$. This example serves as a reaffirmation of the theory.

2.2 Lagurre polynomials

The weight function $w(x) = \mathrm{e}^{-x}\chi_{[0,\infty)}(x)$ which induces the Laguarre orthogonal polynomial sequence gives us

$$\varphi_n(x) = i^n\sqrt{\frac{2}{\pi}}\left(\frac{1+2ix}{1-2ix}\right)^n\frac{1}{1-2ix}. \tag{4}$$

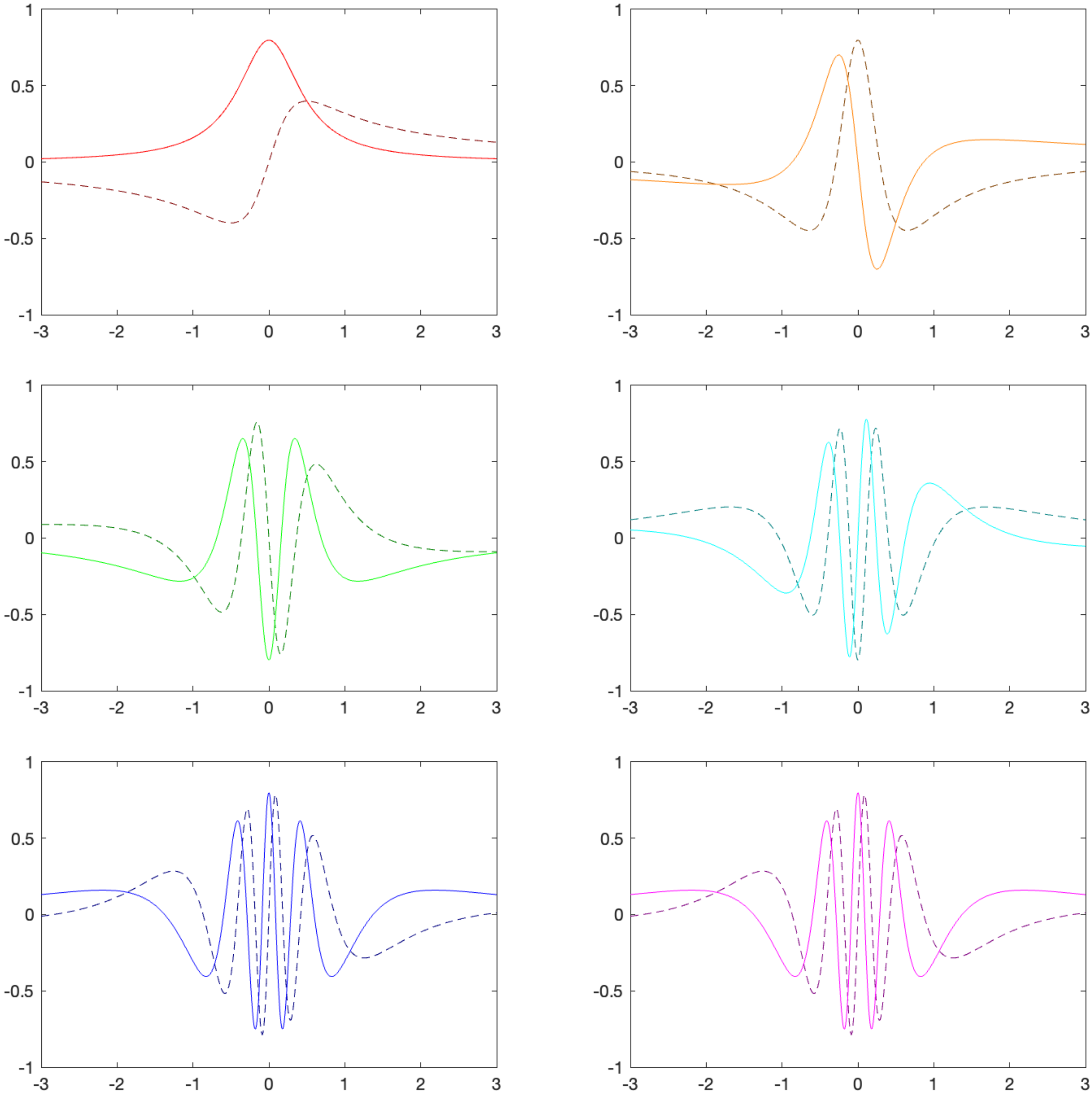

Figure 1: The Malmquist-Takenaka Functions for $n = 0, \ldots, 5$; solid/dotted lines represent real/imaginary part respectively. All functions are of the form $x \mapsto i^n \exp(ig_n(x))/(1-2ix)$, where $g_n : \mathbb{R} \to (-n\pi/2, 3n\pi/2)$ is monotonic.

These functions already exist in the literature and are known as the *Malmquist-Takenaka functions*. In order for our basis to span $L_2(\mathbb{R})$ it is neccesary that the support of our weight function is the whole real line (remember, we have no hope of our basis spanning $L_2(\mathbb{R})$ if the Fourier transform of our basis functions don't span $L_2(\mathbb{R})$ either). Therefore we consider $\{\varphi_n\}_{n\in\mathbb{Z}} := \{\varphi_n\}_{n=0}^{\infty} \sqcup \{\varphi_n\}_{n=-\infty}^{-1}$ where for negative n we have φ_n is given by eq. 3 with $p_n = \tilde{p}_{-n}$ and $\{\tilde{p}_m\}_{m=0}^{\infty}$ is the orthogonal polynomial sequence induced by the mirrored weight function $\tilde{w}(x) := w(-x)$. Then by proposition 1.3 we have that $\{\varphi_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis[2] of $L_2(\mathbb{R})$. In fact it turns out our new φ_n with negative index follow eq. 4, so we can say eq. 4 is true for all $n \in \mathbb{Z}$, we refer to this as the MT basis.

An unignorable fact about the MT basis is the ability to rewrite the coefficients of a function in the Fourier basis, which allows the use of a FFT. Using subsitution $\mathrm{e}^{i\theta} = \frac{1+2ix}{1-2ix}$ we get

$$\int_{-\infty}^{\infty} f(x)\overline{\varphi_n(x)}\mathrm{d}x = \frac{(-i)^n}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \left(1 - i\tan\frac{\theta}{2}\right) f\left(\frac{1}{2}\tan\frac{\theta}{2}\right) \mathrm{e}^{-in\theta}\mathrm{d}\theta.$$

2.3 Continuous Hahn polynomials

The continuous Hahn polynomials have the weight function $w_{a,b}(x) = \frac{1}{2\pi}|\Gamma(a+ix)\Gamma(b-ix)|^2$ where the complex parameters a, b have positive real part. From lemma 2.1 in [Koe94] we have[3]

$$\varphi_n^{(a,b)}(x) = (1-\tanh x)^a (1+\tanh x)^b \mathrm{P}_n^{(2a-1,2b-1)}(\tanh x) \quad n = 0, 1, 2, \ldots. \tag{5}$$

Where $\mathrm{P}_n^{(\alpha,\beta)}$ are the standard Jacobi polynomials. Note here that the weight function is supported on the whole real line, so we know by proposition 1.3 that Φ is indeed a T-system.

Looking at corresponding family of weight functions $w_{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$ associated with Jacobi polynomials, it is clear that

$$\mathrm{P}_n^{(\alpha,\beta)} \propto \begin{cases} \mathrm{T}_n & (\alpha,\beta) = (-\frac{1}{2}, -\frac{1}{2}), \\ \mathrm{U}_n & (\alpha,\beta) = (\frac{1}{2}, \frac{1}{2}), \\ \mathrm{V}_n & (\alpha,\beta) = (-\frac{1}{2}, \frac{1}{2}), \\ \mathrm{W}_n & (\alpha,\beta) = (\frac{1}{2}, -\frac{1}{2}), \end{cases}$$

where $\mathrm{T}_n, \mathrm{U}_n, \mathrm{V}_n, \mathrm{W}_n$ are the standard first, second, third, fourth Chebyshev polynomials respecetively. This motivates us to write out the expansion formulæ in the Fourier basis.

Writing $\varphi_n^{(a,b)} = \tau_n, \upsilon_n, \phi_n, \psi_n$ for $(a,b) = (\frac{1}{4}, \frac{1}{4}), (\frac{3}{4}, \frac{3}{4}), (\frac{1}{4}, \frac{3}{4}), (\frac{3}{4}, \frac{1}{4})$, we get

[1] When we say Φ is orthonormal, it is in this sense which we mean it.

[2] Note that $\tilde{p}_0 = p_0$, so in particular $\{\varphi_n\}_{n=0}^{\infty} \sqcup \{\varphi_n\}_{n=-\infty}^{-1} = \{\varphi_n\}_{n=0}^{\infty} \cup \{\varphi_n\}_{n=-\infty}^{0}$ which has span $\mathrm{PW}_{[0,\infty)}(\mathbb{R}) + \mathrm{PW}_{(-\infty,0]}(\mathbb{R}) = L_2(\mathbb{R})$.

[3] Here $\mathrm{P}_n^{(\alpha,\beta)} = \frac{(\alpha+1)_n}{n!} {}_2F_1(-n, 1+\alpha+\beta+n; \alpha+1; \frac{1}{2}(1-z))$ are the Jacobi polynomials generated by weight function $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$.

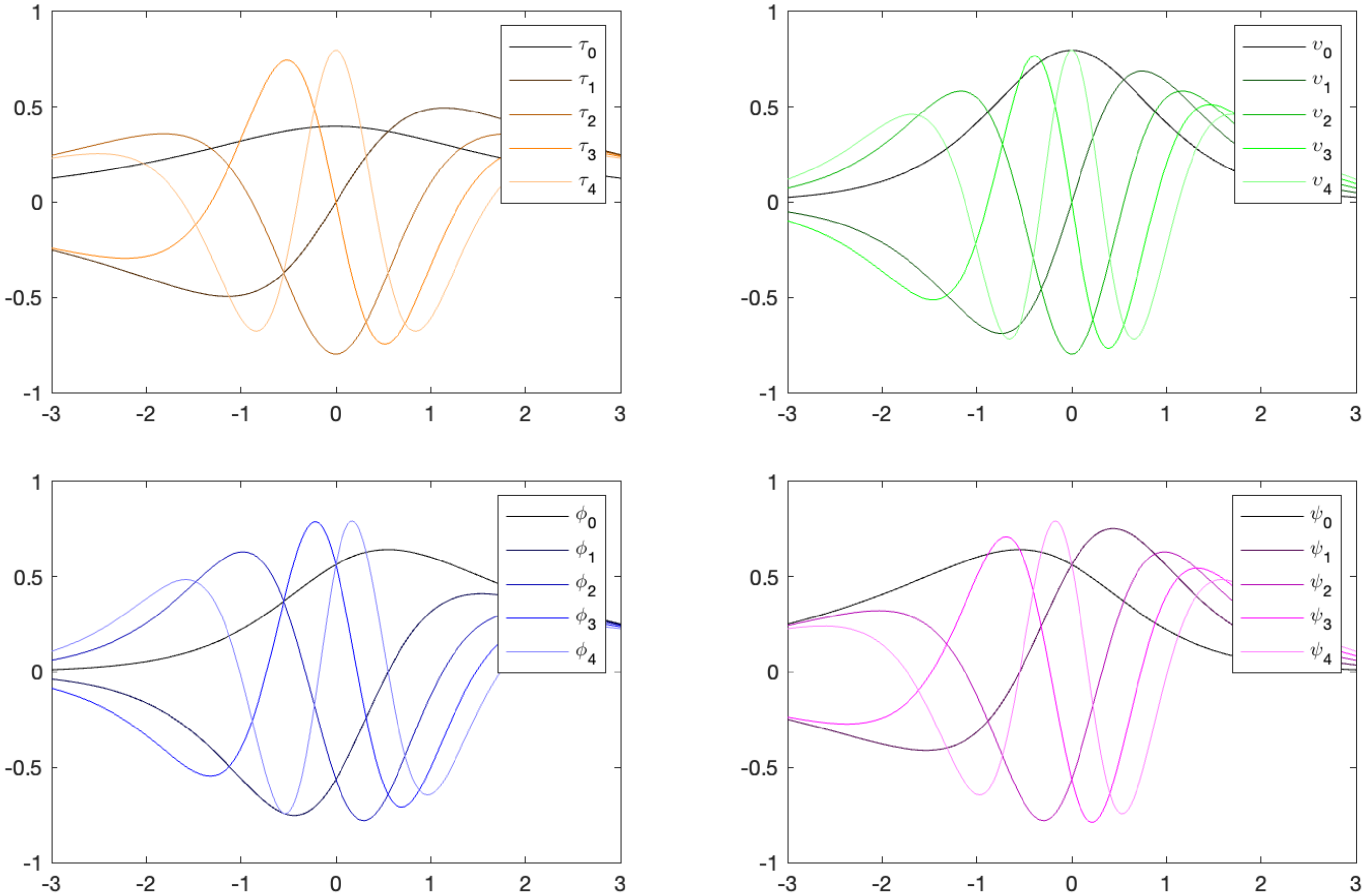

Figure 2: The *normalised* transformed Continuous Hahn Polynomials.

four seperate T-systems. More concretly we have

$$\begin{aligned}\tau_n(x) &= \sqrt{\tanh x}\mathrm{T}_n(\tanh x)\\ \upsilon_n(x) &= \sqrt{\tanh^3 x}\mathrm{U}_n(\tanh x)\\ \phi_n(x) &= \sqrt{\tanh x(1+\tanh x)}\mathrm{V}_n(\tanh x)\\ \psi_n(x) &= \sqrt{\tanh x(1-\tanh x)}\mathrm{W}_n(\tanh x).\end{aligned}$$

By reparameterising with $\tanh x = \cos\theta$ we get expansion formulæ

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\tau_n(x)\mathrm{d}x &= \int_0^{\pi} f\left(\log\cot\frac{\theta}{2}\right)\frac{\cos n\theta}{\sqrt{\sin\theta}}\mathrm{d}\theta\\ \int_{-\infty}^{\infty} f(x)\upsilon_n(x)\mathrm{d}x &= \int_0^{\pi} f\left(\log\cot\frac{\theta}{2}\right)\frac{\sin(n+1)\theta}{\sqrt{\sin\theta}}\mathrm{d}\theta\\ \int_{-\infty}^{\infty} f(x)\phi_n(x)\mathrm{d}x &= \sqrt{2}\int_0^{\pi} f\left(\log\cot\frac{\theta}{2}\right)\frac{\cos(n+1/2)\theta}{\sqrt{\sin\theta}}\mathrm{d}\theta\\ \int_{-\infty}^{\infty} f(x)\psi_n(x)\mathrm{d}x &= \sqrt{2}\int_0^{\pi} f\left(\log\cot\frac{\theta}{2}\right)\frac{\sin(n+1/2)\theta}{\sqrt{\sin\theta}}\mathrm{d}\theta\end{aligned}$$

which allows us the use of a fast cosine transform.

3 On the rate of convergence of T-systems

The main questions which remain to be answered is *'what is the theory of the convergence of orthogonal series on the real line?'*. Unfortunately there is no satisfactory theory going beyond rational functions. There is an excellent theory on the convergence on compact intervals; one might even try to compactify the real line and adapt the Bernstein ellipse, however, this immediatly fails since non-rational analytic functions *neccesarily* have a singularity at infinity. So even though we can 'squeeze' our function into a compact basis to use FFT/FCT, the best we get is that the coefficients are $O(1)$[4].

In [ILW23] an ad-hoc detailed account of using the MT basis to approximate wave-packets is given.

References

[ILW23] Arieh Iserles, Karen Luong, and Marcus Webb. Approximation of wave packets on the real line. *Constructive Approximation*, 58(1):199–250, 2023.

[IW21] Arieh Iserles and Marcus Webb. *A Differential Analogue of Favard's Theorem*, pages 239–263. Springer International Publishing, Cham, 2021.

[Koe94] Erik Koelink. On jacobi and continuous hahn polynomials. *Proceedings of the American Mathematical Society*, 124, 09 1994.

[4]This is laughably bad noting that the Riemann-Lebesgue lemma tells us that in the case of MT functions we get at least $o(1)$.