INTRODUCTION TO SCHEMES

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Contents

0. Preamble 3
0.1. Prerequisites 3
0.2. Other Notes 3
0.3. Some Books 3
1. Why schemes? 3
1.1. Summary of affine varieties 3
1.2. Morphisms between affine varieties 4
1.3. Why varieties are not good enough? 4
1.4. Intuition 4
1.5. Historical motivation (non-examinable) 4
2. The prime spectrum 6
2.1. Generic points 7
2.2. Topology of $\operatorname{Spec} R$ 9
2.3. Morphisms between spectra 9
3. Sheaves 10
3.1. Preliminary definitions 10
3.2. Stalks 13
3.3. Kernels and Cokernels 13
3.4. Moving between spaces 14
4. Affine schemes 16
4.1. Structure sheaf 16
4.2. Affine schemes 19
5. Schemes 22
5.1. Integral scheme 23
6. Fibre products and all that jazz 25
6.1. Fibre products 26
6.2. Examples of fibre products 28
6.3. Separatedness 29
6.4. Varieties 29
7. Morphisms 30
7.1. Properness 30
7.2. Valuative criterion 31
7.3. Flatness 32
8. Sheaves of modules 34
8.1. $\mathcal{O}_X$-modules 34

Date: Hilary Term 2025.

8.2. (Quasi-)coherent sheaves 34
8.3. Properties of (quasi-)coherent sheaves 36
8.4. Vector bundles 37
8.5. Why vector bundles are called so? 38
9. Divisors 40
9.1. Weil divisors 40
9.2. Calculations 41
9.3. Cartier divisors 41
9.4. Cartier to Weil 42
10. Čech cohomology 43
10.1. Definition and examples 43
10.2. Cohomology of affine schemes 45
10.3. A long exact sequence on H^* 46
10.4. Cohomology of $\mathbb{P}^r$ 47
11. Cohomology, divisors and miracles 49
11.1. Pic, CaCl, and H^1 went to a party 49
11.2. Satz von Riemann–Roch 50

FOREWARD

These are a series of lecture notes based on the lecture series Maria Yakerson lectured in Hilary term 2024 for the course C2.6 Introduction to Schemes course at the University of Oxford. I claim no originality whatsoever and any typos/mistakes are most definitely mine.

LECTURE 1

0. PREAMBLE

0.1. Prerequisites.

- Commutative algebra (Atiyah MacDonald)
- Category theory and homological algebra (not much)
- C3.4 Algebraic Geometry (intuition & ideas)

0.2. Other Notes.

- Lecture notes (online)
- A. Ritter HT 2020-22
- D. Rangauathan AG Part III Cambridge
- Introduction to schemes G Ellingrand. J. Otten

0.3. Some Books.

1. The rising sea
2. Algebraic geometry and arithmetic curves
3. D. Eisenbud, J. Harris 'The geometry of schemes'
4. Stacks project (precise & detailed)

1. WHY SCHEMES?

1.1. **Summary of affine varieties.** k-algebra closed field Main idea:

$$\begin{aligned} &\{\text{subsets of } k^n \text{ cut out by polynomial equations}\} \\ \leftrightarrow &\{\text{finitely generated } k\text{-algebras without nilpotent elements}\} \end{aligned}$$

$$\text{geometry} \simeq \text{algebra}$$

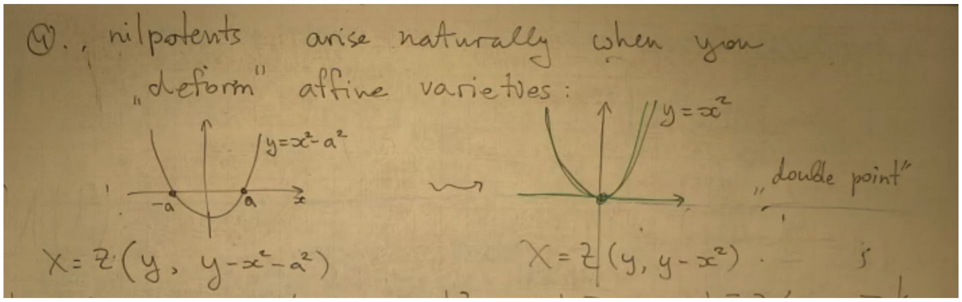
- $I \lhd k[x_1, \dots, x_n]$ ideal
- $\mathbb{A}^n \supset X := Z(I) = \{a \in k^n : f(a) = 0 \forall f \in I\}$ affine variety
- $\mathbb{A}^n(k) =: \mathbb{A}^n$ - n-dimensional *affine space*, set: k^n
- Zariski topology: closed subsets are $Z(I)$. Basis of Zariski topology: $D(f) = \{a : f(a) \neq 0\}$. Any $X \subset \mathbb{A}^n$ has the subspace topology
- $I(X) := \{f \in k[x_1, \dots, x_n] : f(x) \forall x \in X\}$, $k[X] := k[x_1, \dots, x_n]/I(x)$ - coordinate ring of X.
- $k[X]$ parameterises functions on X: $x \in X \to \mathfrak{m} := \ker(\mathrm{ev}_x : k[X] \to k)$ and for all $f \in k[X]$ gives $f : X \to \mathbb{A}^1 (= k)$

Proposition 1.1 (Hilbert's Weak Nullstellensatz).

$$\begin{aligned} \{\textit{points of } X\} &\leftrightarrow \{\textit{maximal ideals of } k[X]\} \\ (a_1, \dots, a_n) &\leftrightarrow \{\overline{x_1 - a_1}, \dots, \overline{x_n - a_n}\} \end{aligned}$$

Proposition 1.2 (Hilbert's Nullstellensatz).

$$I(Z(I)) = \sqrt{I} =: \{f \in I : f^m \in I \textit{ for some } m \in \mathbb{N}\}$$



④. nilpotents arise naturally when you „deform" affine varieties:

Figure 1. Nilpotency

1.2. **Morphisms between affine varieties.** Given $X, Y \subset \mathbb{A}^m$, a morphism of varieties is given by

$$(f_1, \dots f_m) = \varphi : X \to Y \subset \mathbb{A}^m$$

where $f_i \in k[X]$ whose image lives in Y. That's equivalent to a *pullback map*

$$\varphi^* : k[X] \to k[Y]$$

so $\mathrm{Hom}(X, Y) = \mathrm{Hom}(k[Y], k[X])$, implies equivalence of categories

$$\text{affine varieties}/k \simeq \text{finitely reduced algebras}^{\mathrm{op}} \tag{1.1}$$

1.3. **Why varieties are not good enough?**

1. Embedding into $\mathbb{A}^n$ shouldn't really be part of the data - would be nice to have an intrinsic definition.
2. When $k \neq \overline{k}$, - Nullstellensatz doesn't work: $I := (x^2 + y^2 + 1) \subset \mathbb{R}[x, y]$ is prime so it's radical but $Z(I) = \emptyset$, hence $I(Z(I)) = \mathbb{R}[x, y]$
3. Question: what is $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$ naturally the space of functions on? Or $\mathbb{R}[x]$? Or $\mathbb{Z}[x]$? Or $\mathbb{Z}$? *Why not take all rings?*
4. Nilpotents arise naturally when you 'deform' affine varieties.$X = Z(y, y - x^2 - a^2)$, $X = Z(y, y - x^2)$. $k[X] = k[x]/(x - a) \oplus k[x]/(x + a) \simeq k^2$...parameterises values at $\{a\}$ and $\{-a\}$; $k[X] = k[x]/\sqrt{(x)} = k$...we lost information because we didn't distinguish x and x^2. We'd like $k[X]' =' k[x]/x^2$.

1.4. **Intuition.** Intersections of varieties often don't want to be varieties!

1.5. **Historical motivation (non-examinable).** Weil conjectures (1949)

- f homogeneous polynomial in $\mathbb{Z}[x_1, \dots, x_n]$
- $X = Z(f) \subset \mathbb{P}^n$ projective hypersurface
- $X(\mathbb{C})$ compact topological space → $b_0(X), \dots, b_{2n}(X)$ betti numbers of X; $b_i := \dim H^i(X(\mathbb{C}); \mathbb{Z})$
- $|X(\mathbb{F}_p)| =: N_m$ - number of solutions reduction modulo $p \to \zeta(X; t) := \exp\left(\sum \frac{N_m}{m} t^m\right)$ Weil Zeta function

Theorem 1.3. *Theorem (Grothedieck, Serre, Artin, Deligne,...) X smooth over $\mathbb{C}$ and over $\overline{\mathbb{F}_p}$, then $\zeta(X;t)$ is a rational function:*

$$\zeta(X;t) = \frac{p_1(t)p_3(t)\cdots p_{2n+1}(t)}{p_0(t)\cdots p_{2n}(t)} \tag{1.2}$$

and

$$\underbrace{\deg p_i(t)}_{arithmetic} = \underbrace{b_i}_{topology}$$

Schemes and cohomology of sheaves were invented for this purpose!

Lecture 2

2. The prime spectrum

Before:

$$(\text{affine varieties over } k = \overline{k})^{\text{op}} \simeq (\text{reduced f.g. algebras over } k = \overline{k})$$

Now:

$$(\text{affine schemes})^{\text{op}} \simeq (\text{rings}) \quad (\text{ass. comm. with } 1)$$

Allows: arithmetic phenomena by geometric methods (rings: $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathcal{O}_K$, etc). Recall: X affine variety over $k = \overline{k}$ implies by Nullstellensatz that

$$\text{points} \leftrightarrow \text{max ideals } \mathfrak{m} = \{f \in k[X] : f(x) = 0\}_x \lhd k[X]$$

Definition 2.1. Let R be a ring. Its *(prime) spectrum* is

$$\operatorname{Spec} R := \{\mathfrak{p} : \mathfrak{p} \lhd R \text{ prime}\}$$

N.B.: we cannot think of $f \in R$ as function with value in some k.

Definition 2.2. Let $x \in \operatorname{Spec} R$ correspond to $\mathfrak{p} \lhd R$. The *residue field* of x (or $\mathfrak{p}$) is

$$\kappa(x) = \kappa(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \quad \text{- a field}$$

Every element $f \in R$ has a 'value'

$$f(x) := f \mod \mathfrak{p}_x \in \kappa(x) \quad \forall x \in \operatorname{Spec} R$$

Moral: $\operatorname{Spec} R$ will be the space on which R is the ring of functions.

Definition 2.3 (The Zariski topology on $\operatorname{Spec} R$)**.** The closed sets are perscribed as

$$Z(\mathfrak{a}) := \{x \in \operatorname{Spec} R : f(x) \quad \forall f \in \mathfrak{a}\} = \{\mathfrak{p} \lhd \text{ prime} : \mathfrak{p} \supset \mathfrak{a}\}$$

where $\mathfrak{a} \lhd R$.

Proposition 2.4. *Let* $\mathfrak{a}, \mathfrak{b} \lhd R$. *Then:*

1. $Z(\mathfrak{a}) \subset Z(\mathfrak{b})$ *if and only if* $\sqrt{\mathfrak{a}} \supset \sqrt{\mathfrak{b}}$. *In particular* $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$
2. $Z(\mathfrak{a}) = \emptyset$ *if and only if* $\mathfrak{a} = R$; $Z(\mathfrak{a}) = \operatorname{Spec} R$ *if and only if* $\mathfrak{a} \subset \sqrt{0} := \operatorname{Nil} R$
3. $Z(\mathfrak{a}) \cup Z(\mathfrak{b}) = Z(\mathfrak{a} \cap \mathfrak{b})$; $\bigcap_{\alpha \in A} Z(\mathfrak{a}_\alpha) = Z(\sum_{\alpha \in A} \mathfrak{a}_\alpha)$

Proof. Use the main fact that $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \supset \mathfrak{a}} \mathfrak{p}$ ☐

Corollary 2.5. *There exists an inclusion reversing bijection*

$$\textit{closed subsets of } \operatorname{Spec} R \leftrightarrow \textit{radical ideals of } R$$

$$Z(\mathfrak{a}) \leftarrow\!\shortmid \mathfrak{a}$$

$$Z \mapsto I(\mathcal{Z}) := \bigcap_{\mathfrak{p} \in Z} \mathfrak{p} = \{f \in R : f(x) \quad \forall x \in Z\}$$

Corollary 2.6. *The closure of any subset* $S \subset \operatorname{Spec} R$ *is of the form* $\overline{S} = Z(a)$ *where* $\mathfrak{a} = \bigcap_{\mathfrak{p} \in S}$. *In particular, for* $S = \{\mathfrak{p}\}$ *we get*

$$\overline{\{\mathfrak{p}\}} = Z(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} R : \mathfrak{q} \supset \mathfrak{p} \textit{ prime}\}$$

Corollary 2.7. $x \in \operatorname{Spec} R$ *is closed if and only if* $\mathfrak{p}_x$ *is a* maximal *ideal.*

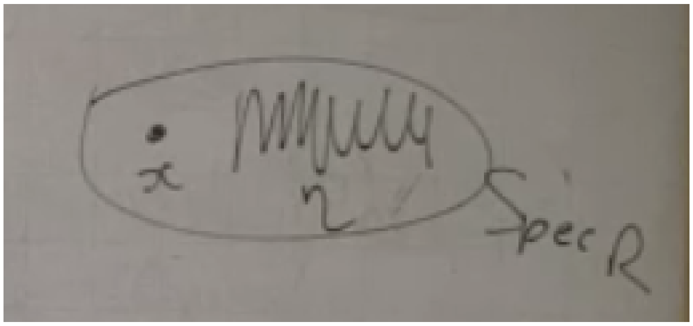


FIGURE 2. Generic point

N.B.: points don't have to be closed! Moral: X variety implies

$$X \simeq \operatorname{mSpec} k[X] \subset \operatorname{Spec} k[X]$$

Motivation: why prime ideals instead of maximal? For varieties over $k = \overline{k}$, Nullstellensatz followed from the *Jacobson property* of f.g. reduced $k = \overline{k}$-algebras:

$$\sqrt{I} = \bigcap_{\mathfrak{m} \supset I} \quad \forall I \subset k[X]$$

and that led to the bijection between closed subsets of X and radical ideals of $k[X]$. For a general R we must use prime ideals to get such a correspondence: If R is a discrete valuation ring (dvr), then there exists a unique maximal ideal $\mathfrak{m} = (t) \subset R$, but R has two radical ideals: (t) and (0), so maximal ideals would not be enough.

2.1. **Generic points.**

Definition 2.8. Let X be a topological space, $Z \subset X$ a closed subset. A generic point of Z (if it exists) is a point $\eta \in Z$ such that $\overline{\{\eta\}} = Z$, i.e., η is a dense point.

In our context:, each $\mathfrak{p} \in \operatorname{Spec} R$ is a generic point of $Z(\mathfrak{p}) \subset \operatorname{Spec} R$.

Example 2.9 (Main example)**.** Let R be an integral domain, then $\mathfrak{p} = (0)$ is *the* generic point of $\operatorname{Spec} R$

Remark 2.10. We'll see that for $X = \operatorname{Spec} R$, that any closed subset $Z \subset X$ has a unique generic point.

Example 2.11. (1) If $R = K$ is a field, then $\operatorname{Spec} K = \{(0)\}$
(2) If $R = K[t]/(t^n)$ ('thickening'), then $\operatorname{Spec} R = \{(t)\}$
1) vs 2): Same topological spaces but with different algebraic structures
(3) If R is an Artinian ring, then $\operatorname{Spec} R$ is a finite set
(4) If R is a dvr, then $\operatorname{Spec} R = \{x, \eta\}$ where x the closed maximal ideal, and η is the open generic point.
(5) If $R = \mathbb{Z}$, then $\mathfrak{p} \in \operatorname{Spec} R$ implies

$$\mathfrak{p} = \begin{cases} (0) & \text{- the generic point} \\ (p) & p \text{ prime - closed point} \end{cases}$$

- $\kappa(\mathfrak{p}) = \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} = \mathbb{F}_p$
- $\kappa(0) = \mathbb{Z}_{(0)} = \mathbb{Q}$

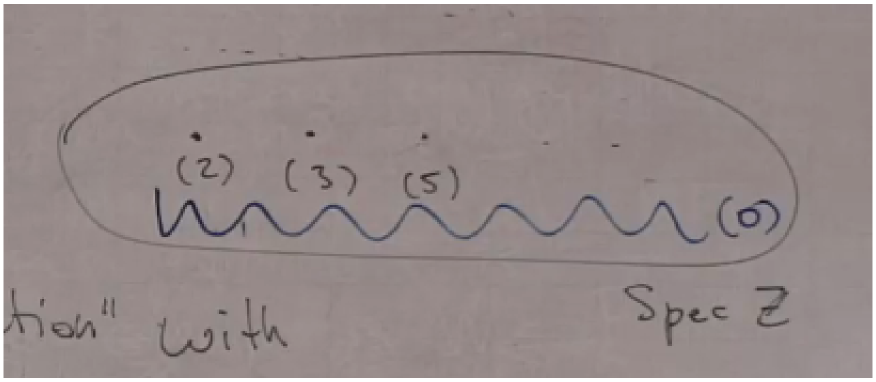


Figure 3. Spec $\mathbb{Z}$

Every $f \in \mathbb{Z}$ gives a 'function' with values in various fields: let $f = 17 \in \mathbb{Z}$, then

$$\begin{aligned} f((0)) &= 17 \in \mathbb{Q} \\ f((2)) &= \overline{1} \in \mathbb{F}_2 \\ f((3)) &= \overline{2} \in \mathbb{F}_3 \\ f((5)) &= \overline{2} \in \mathbb{F}_5 \\ &\vdots \end{aligned}$$

Comments:

1. When R is a finitely generated k-algebra over $k = \overline{k}$, then for any closed point $\mathfrak{m} \in \operatorname{Spec} R$ we have $\kappa(\mathfrak{m}) = k$ by the Nullstellensatz which says $\kappa(\mathfrak{m})/k$ is a finite field extension.
2. For such R, the topology of $\operatorname{Spec} R$ is fully detected by closed points, but the diversity of residue fields allows:... and to prove Fermat's last theorem:

Definition 2.12. The affine n-space is

$$\mathbb{A}^n := \operatorname{Spec} \mathbb{Z}[t_1, \dots, t_n]$$
$$\mathbb{A}^n_R := \operatorname{Spec} R[t_1, \dots, t_n]$$

If $k = \overline{k}$:

$$\mathbb{A}^n_k \supset \mathbb{A}^n(k) = k^n$$
$$\text{prime ideals} \supset \text{maximal ideals}$$

The Zariski topology is induced here.

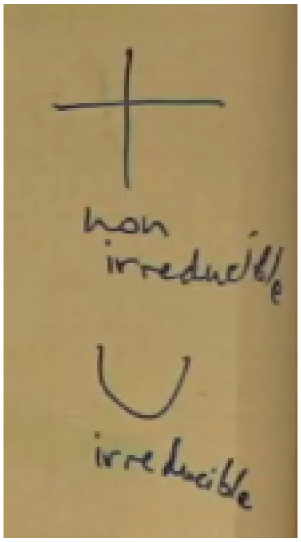


Figure 4. Irreducibility visualised

Lecture 3

Chapter 2 continues

2.2. Topology of Spec R.

From last time:

Definition 2.13. A distinguished open set in $X = \operatorname{Spec} R$ is

$$D(f) := X - Z(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\} \quad \forall f \in R$$

Lemma 2.14.
1. *$D(f) = \emptyset$ if and only if $f \in R$ is nilpotent*
2. *$D(f) \cap D(g) = D(f \cdot g)$*
3. *$D(g) \subset D(f)$ if and only if $g^n \in (f)$ for some $n \in \mathbb{N}$*
4. *$\{D(f)\}_{f \in R}$ is a basis for the Zariski topology on* Spec R
5. *$\bigcup_{i \in I} D(f_i) = \operatorname{Spec} R$ if and only if $1 = \sum_{j=1}^{N} a_{ij} f_j$ for some $a_{i1}, \ldots, a_{iN} \in R$*

In particular, Spec R *is quasi-compact.*

We can describe algebraically the irreducibility of closed subsets.

Proposition 2.15.
1. *$\mathfrak{p} \in \operatorname{Spec} R$ implies that overline$\{\mathfrak{p}\} = Z(\mathfrak{p})$ and $\{\mathfrak{p}\}$ is the only generic point of $Z(\mathfrak{p})$.*
2. *$Z \subset \operatorname{Spec} R$ is irreducible if and only if $Z = Z(\mathfrak{p})$ for some $\mathfrak{p} \in \operatorname{Spec} R$.*
3. *$\operatorname{Spec} R$ is irreducible if and only if $\operatorname{Nil} R := \sqrt{(0)}$ is prime.*

Corollary 2.16. *A nonempty irreducible subspace $Z \subset \operatorname{Spec} R$ has a unique generic point.*

Proposition 2.17. *Let R be noetherian. If $Z \subset \operatorname{Spec} R$ is a closed subset, then $Z = Z_1 \cup \cdots \cup Z_r$ for some unique closed irreducible $Z_i \subset \operatorname{Spec} R$ (up to reordering).*

2.3. Morphisms between spectra.

Yet another reason why we need $\mathfrak{p}$ not in: $\varphi : R \to S$ and $\mathfrak{m} \lhd S$ implies $\varphi^{-1}(\mathfrak{m})$ doesn't have to be a maximal ideal. Luckily $\varphi^{-1}(\mathfrak{m})$ is always prime!

Example 2.18. The inclusion morphism $i : k[x] \hookrightarrow k(x)$ induces a map of spectra $\operatorname{Spec} k(x) \to \operatorname{Spec} k[x]$ such that any closed point gets mapped to a generic point.

Proposition 2.19. *There's a contravariant functor*

$$\begin{aligned} \operatorname{Spec} : \mathbf{Ring}^{\mathrm{op}} &\to \mathbf{Top} \\ R &\mapsto \operatorname{Spec} R \\ (\varphi : R \to S) &\mapsto [\varphi^* : \operatorname{Spec} S \to \operatorname{Spec} R; \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})] \end{aligned}$$

Proposition 2.20. *Let $\varphi : R \to S$ be a ring homomorphism and let $\Phi = \operatorname{Spec} \varphi$.*

(1) *If φ is surjective, then $\Phi : \operatorname{Spec} S \xrightarrow{\sim} Z(\ker \varphi) \subset \operatorname{Spec} R$*
(2) *If φ is injective, then $\Phi(\operatorname{Spec} S)$ is a dense subset. Moreover, $\operatorname{Im} \Phi \subset \operatorname{Spec} R$ is dense if and only if $\ker \phi \subset \operatorname{Nil} R$*

Example 2.21. (1) (Quotients) Let $\mathfrak{a} \lhd R$, then

$$\begin{tikzcd} \operatorname{Spec} R/\mathfrak{a} \arrow[r] \arrow[dr, "\sim"] & \operatorname{Spec} R \arrow[d] \\ & Z(\mathfrak{a}) \end{tikzcd}$$

(2) (Localisations) Let $f \in R$

$$\begin{tikzcd} \operatorname{Spec} R_f \arrow[r] \arrow[dr, "\sim"] & \operatorname{Spec} R \arrow[d] \\ & D(f) \end{tikzcd}$$

(3) (Reductions)

$$\begin{tikzcd} \operatorname{Spec} \mathbb{F}_p \arrow[r] \arrow[dr, "\sim"] & \operatorname{Spec} \mathbb{Z} \arrow[d] \\ & \{(\mathfrak{p})\} \end{tikzcd}$$

More generally, since $\mathbb{Z}$ is an initial object in **Ring**, there is a map $\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}$ and it factors through $\operatorname{Spec} \mathbb{F}_p$ if and only if R is of character p.

3. Sheaves

3.1. **Preliminary definitions.** Main idea: a scheme is a space that *locally* looks like $\operatorname{Spec} R$ with 'functions' on it.

Definition 3.1. A *presheaf* of sets (groups, rings, spaces,...) on a category $\mathcal{C}$ is a functor

$$\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}/\mathbf{Grp}/\mathbf{Ring}/\dots$$

A *presheaf on a topological space* X is a presehaf on $\mathbf{Open}(X)$:

$$\begin{aligned} \mathrm{Obj} &= \text{open subset } U \subset X \\ \mathrm{Mor} &= \text{inclusions of open sets} \end{aligned}$$

That is, a presheaf R on X consists of:

$$\begin{aligned} U &\mapsto R(U) \quad \text{(set/group/ring...)} \\ (V \hookrightarrow U) &\mapsto (\rho_{UV} : R(U) \to R(V)) \quad \text{(map/hom/...)} \end{aligned}$$

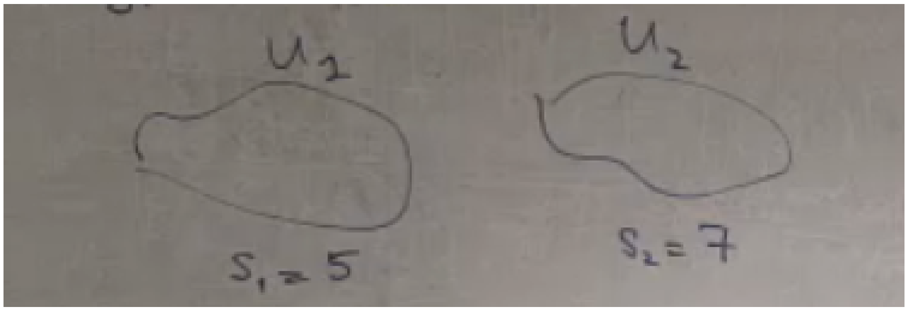


Figure 5. The failure of the constant presheaf of being a sheaf

such that $\rho_{UU} = \mathrm{id}_{R(U)}$ and $\rho_{UW} = \rho_{UV} \circ \rho_{VW}$ for $U \supset V \supset W$. The elements of $R(U)$ are called *sections* and the elements of $R(X)$ are called *global sections*. We write $\rho_{UV}(f) = f|_V$.

Example 3.2.

1. The constant presehaf A_X on X is specified by picking and A and setting
$$A_X(U) = A \quad \forall U$$
$$\rho_{UV} = \mathrm{id}_A \quad \forall V \subset U$$
2. The presehaf of C^∞-functions on a smooth manifold X is defined by $R(U) := C^\infty(U;\mathbb{R})$ for all U, and the ρ_{UV} are restrictions of functions. Want: glue values on X from local data

Definition 3.3. A *sheaf* R on X is a presheaf on X such that

1. For all open covers of a subset $U = \bigcup_i U_i \subset X$ and $s, t \in R(U)$, if $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$
2. If $U = \bigcup_i U_i \subset X$ is an open cover, and $s_i \in R(U_i)$ is a collection of sections with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists an $s \in R(U)$ such that $s|_{U_i} = s_i$ for all i.

Remark 3.4. $\mathscr{F}(\emptyset) = *$ is a terminal object.

The constant preseaf is *not* a sheaf: say $X = U_1 \coprod U_2$ with $A = \mathbb{Z}$ $s_i \in A_X(U_i)$, $s_1|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2}$ because $U_1 \cap U_2 = \emptyset$ but there does not exist an $s \in A_X(X) = \mathbb{Z}$ such that $s|_{U_i} = s_i$ because restriction maps are identities.

Fix this: the *constant sheaf*:

$$A_X(U) := \{\text{locally-constant } U \to A\} = \prod_{\Gamma \in \pi_0 U} A$$

It's an example of *sheafification*: for any presheaf $\mathscr{F}$ and sheaf $\mathscr{G}$ and any morphism of presheaves $\mathscr{F} \to \mathscr{G}$ there exists a sheaf $\mathscr{F}^+$ such that

$$\begin{tikzcd}
\mathscr{F} \arrow[r] \arrow[d] & \mathscr{G} \\
\mathscr{F}^+ \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

Definition 3.5. A morphism between (pre)sheaves is a natural transformation of functors.

Definition 3.6. A sub(pre)sheaf $\mathscr{F} \subset \mathscr{G}$: is such that $\mathscr{F}(U) \subset \mathscr{G}(U)$ for all open sets U.

Lecture 4

3.2. **Stalks.**

Definition 3.7. Let $\mathscr{F}$ be a (pre)sheaf on X and let $x \in X$. The *stalk of* $\mathscr{F}$ *at* x is:

$$\mathscr{F}_x := \operatorname{colim}_{x \in U \subset X} \mathscr{F}(U) \quad \text{colimit with respect to restriction maps}$$

Explicitly: each element of $\mathscr{F}_x$ is determined by $f \in \mathscr{F}(U)$ with $U \subset X$ open and $(f, U) \sim (f', U')$ if there exists an open subset $W \subset U \cap U'$ with $x \in W$ such that $f|_W = f'|_W$. The class of equivalence of (f, U) in $\mathscr{F}_x$ is the *germ* of f at x.

Remark 3.8.
1. $\mathscr{F}_x$ has the same algebraic structure as $\mathscr{F}$ (group/ring/…)
2. 'Stalks encode local data'
3. For all $x \in U$, $\mathscr{F}_x \simeq (\mathscr{F}|_U)_x$
4. $\mathscr{F}_x \simeq \mathscr{F}_x^+$ for all $\mathscr{F}, x$
5. A morphism $\mathscr{F} \to \mathscr{G}$ on X induces $\mathscr{F}_x \to \mathscr{G}_x$

Exercise 3.9 (Stalks are powerful!)**.** Let $\mathscr{F} \to \mathscr{G}$ be a morphism of abelian sheaves on X. This morphism is an isomorphism if and only if the induced maps on stalks are all isomorphisms.

3.3. **Kernels and Cokernels.**

Definition 3.10. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves on X. The presheaf kernel/image/cokernel is

$$U \mapsto \ker(\mathscr{F}(U) \to \mathscr{G}(U))$$

Exercise 3.11. If φ is a map of sheaves, then ker is a sheaf.

Example 3.12 (NOT true for cokernels!)**.** Let $X = \mathbb{C}$, $\mathscr{F}_x :=$ (holomorphic functions on X, $+$), $\mathscr{F}_x^* :=$ (non-zero holomorphic functions on $X, \times$) and $\exp : \mathscr{F}_x \to \mathscr{F}_x^*$. $\ker(\exp) = 2\pi i\underline{\mathbb{Z}}$ constant sheaf. Coker is **not** a sheaf:

$$U_1 = \mathbb{C} \setminus [0, \infty) \quad U_2 = \mathbb{C} \setminus (-\infty, 0] \quad U = U_1 \cup U_2 = \mathbb{C} \setminus 0$$

log exists on each U_i so $\operatorname{coker}(\exp)(U_i) = 0$; however $f = z \in \mathscr{F}_x(U)$ has $\mathscr{F} \neq 0 \in \operatorname{coker}(\exp)(U)$

Definition 3.13. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. The sheaf cokernel/image is the sheafification of coker/im. A morphism is injective/surjective if $\ker\varphi = 0/ \underbrace{\operatorname{im}\varphi}_{\text{sheaf!}} = \mathscr{G}$

Example 3.14. The sequence

$$0 \longrightarrow 2\pi i\underline{\mathbb{Z}} \longrightarrow \mathscr{F}_X \longrightarrow \mathscr{F}_X^\times \longrightarrow 1$$

is *exact* as a sequence of sheaves for all complex manifolds X.

Definition 3.15. Let $\mathscr{F} \subset \mathscr{G}$ be a subsheaf. The *quotient sheaf* is the sheafification of $U \mapsto \mathscr{G}(U)/\mathscr{F}(U)$.

Exercise 3.16.
1. ker and im commute with taking stalks
2. Injectivity and surjectivity are stalk-local properties, but the maps φ_U don't have to be surjective for all U.

3.4. **Moving between spaces.** Let $f : X \to Y$ be a map of topological spaces.

Definition 3.17. The *pushforward* (or *direct image*) $f_*\mathscr{F}$ on Y is the presheaf $U \mapsto \mathscr{F}(f^{-1}(U))$.

Proposition 3.18. *$f_*\mathscr{F}$ is a sheaf.*

Proof. Exercise. $\square$

Definition 3.19. The *inverse image* presheaf is

$$f^{-1}\mathscr{G}^{\text{pre}}(V) := \text{colim}_{U \supset f(V)}\mathscr{G}(U) = \{(s_U, U) : f(V) \subset U \text{ open and } s_U \in \mathscr{G}(U)\}$$

as identifies sections that agree in an open neighbourhood of $f(V)$. The inverse image $f^{-1}(\mathscr{G})$ is its sheafification.

Remark 3.20. The sheafification is necessary like for a constant sheaf:

$$\text{id}_{\text{Y}} \coprod \text{id}_{\text{Y}} : Y \coprod Y \to Y \quad U \subset Y \text{ open}$$

$f^{-1}\mathscr{G}^{\text{pre}}(U \coprod U) = \mathscr{G}(U)$, but for sheaf axioms to hold we will have $f^{-1}\mathscr{G}(U \coprod U) \simeq \mathscr{G}(U) \times \mathscr{G}(U)$.

Example 3.21. (1) $i : S \hookrightarrow X$ open set.

$$\mathscr{F} \in \text{Sh}(S) \qquad\qquad i_*\mathscr{F} : V \mapsto \mathscr{F}(V \cap S)$$

$$\mathscr{G} \in \text{Sh}(S) \qquad i^{-1} : \mathscr{G} : U \mapsto \mathscr{G}(U) \text{ restriction } \mathscr{G}|_S \text{ of } \mathscr{G}$$

(2) $i_x : x \hookrightarrow X$ point with $\mathscr{F} \in \text{Sh}(X)$ and $i_x^{-1}\mathscr{F} = \mathscr{F}_x$.
(3) $\pi : X \to \text{pt}$ with $\mathscr{F} \in \text{Sh}(X)$ and $\pi_*\mathscr{F}_{(\text{pt})} = \mathscr{F}(X) =: \Gamma(X, \mathscr{F})$, the global sections functor.

Proposition 3.22 (f^{-1} is left adjoint to f_*)**.** *There is a natural isomorphism:*

$$\text{Mor}_{\text{Sh}(X)}(f^{-1}\mathscr{G}, \mathscr{F}) \simeq \text{Mor}_{\text{Sh}(X)}(\mathscr{G}, f_*\mathscr{F}).$$

Proof. Sketch: $\boxed{\longrightarrow}$ Given $\text{colim}_{U \supset f(V)}\mathscr{G} \to \mathscr{F}(V)$ for $V \subset X$ open take any $W \subset Y$ open

$$\mathscr{G}(W) \to \text{colim}_{U \supset f(V)}\mathscr{G}(U) \underbrace{\to}_{\text{have}} \overbrace{\mathscr{F}(V)}^{\text{pick } V := f^{-1}W} \quad \to \mathscr{F}(f^{-1}W) = f_*(W).$$

$\boxed{\longleftarrow}$ Given $\mathscr{G}(W) \to \mathscr{F}(f^{-1}W)$ for any open W

$$\begin{tikzcd}
\mathscr{G}(W) \arrow[r] \arrow[d, "\text{assume } W \supset f(V)"'] & \mathscr{F}(f^{-1}W) \arrow[d] & \\
\text{colim}_{U \supset f(V)}\mathscr{G}(U) \arrow[r] & \text{colim}_{U \supset f(V)}\mathscr{F}(f^{-1}U) \arrow[r, "\text{restriction: } f^{-1}U \supset V"] & \mathscr{F}(V)
\end{tikzcd}$$

$\square$

Note 3.23. Let A be a ring, $S \subset A$ a multiplicatively closed subset without zero. Define

$$S^{-1}A := \{(a, s) : s \in S, a \in A\}$$

where $(a, s) \sim (a', s')$ if and only if there exists an $s'' \in S$ such that $s''(as' - a's) = 0$ in A.

Example 3.24.
1. $S = \{1, f, f^2, \ldots\}$ denoted A_f
2. $S = A \setminus \mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal, denoted $A_{\mathfrak{p}}$

Lecture 5

Sheafification doesn't change stalks - categorical proof

$$i : x \hookrightarrow X$$

$$\begin{tikzcd}
\mathrm{Sh}(X) \arrow[r, hook, bend right, "i"] \arrow[d, hook, bend left, "i^{-1}"] & \mathrm{Sh}(X) \arrow[l, bend right, "i^{-1}"] \arrow[d, hook, bend left] \\
\mathrm{Psh}(X) \arrow[u, bend left, "L"] \arrow[r, hook, bend right, "i_*"'] & \mathrm{Psh}(X) \arrow[l, bend right, "i_*^{-1}"] \arrow[u, bend left, "L"]
\end{tikzcd}$$

$$\mathscr{F}_x = i^{-1}\mathscr{F} = Li^{-1}\mathscr{F} = i^{-1}L\mathscr{F} = i^{-1}\mathscr{F}^+ = \mathscr{F}_x^+$$

Finally note that the kernel and cokernel makes sense for sheaves of abelian groups, not sets.

4. Affine schemes

4.1. **Structure sheaf.**

Theorem 4.1. *The structure sheaf* $\mathcal{O}_{\operatorname{Spec} R}$ *is the sheaf of rings on* $\operatorname{Spec} R$ *such that*

1. $\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R_f$ *for all* $f \in R$
2. $\mathcal{O}_{\operatorname{Spec} R,x} = R_{\mathfrak{p}_x}$ *for all* $x \in \operatorname{Spec} R$.

The moral reasons are:

1. $D(f) = \{x \in X : f(x) \neq 0\}$ implies $\mathcal{O}_X(D(f)) = R_f$. We allow to invert powers of f because this won't vanish on $D(f)$.
2. $\mathcal{O}_{X,x} = \{(U, f) : x \in U, f \in \mathcal{O}_X(U)\}/\sim$ then $\mathcal{O}_{X,x} = R_{\mathfrak{p}_x}$ germs encode local behaviour at x; we allow to invert functions that don't vanish at x, i.e., $R_{\mathfrak{p}_x}$.

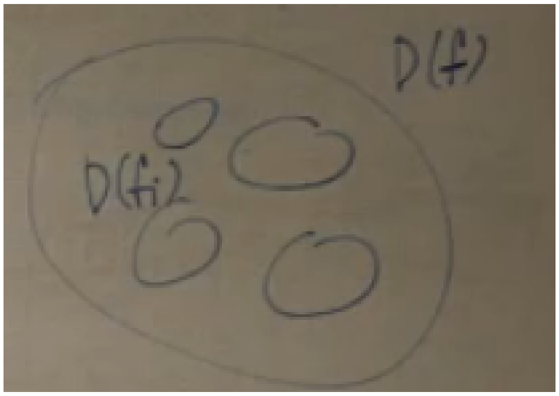
Example 4.2.
1. $X = \operatorname{Spec} \mathbb{Z}$.
   $\mathcal{O}(D(p)) = \mathcal{O}_X(\operatorname{Spec} \mathbb{Z}(p)) = \mathbb{Z}[1/p] = \{m/p^n : m \in \mathbb{Z}, n \geq 0\}$.
   $\mathcal{O}_{X,(p)} = \mathbb{Z}_{(p)} = \{m/\ell : p \not| \ell\}$
   $\mathcal{O}_{X,(0)} = \mathbb{Z}_{(0)} = \mathbb{Q}$
2. $X = \operatorname{Spec} D = \{x, \eta\}$ where D is a dvr, $\mathfrak{m} = (t)$, $K := \mathrm{Frac} D$
   $\mathcal{O}_X(\emptyset) = 0$, $\mathcal{O}_X(X) = D$; $\mathcal{O}_X(\eta) = D_t = K$;
   $\mathcal{O}_{X,x} = D_{(t)} = D$; $\mathcal{O}_{X,\eta} = D_{(0)} = K$.

Proof. $\boxed{\text{I}}$ Define $\mathcal{O}$ as a presheaf on $\{D(f)\}_{f \in R}$ given by $\mathcal{O}(D(f)) = R_f$. Since different f's can give the same $D(f)$, we define $\mathcal{O}(D(f)) := S_{D(f)}^{-1} R$, where

$$\underbrace{S_{D(f)}}_{\text{`saturations of } \{f^n\}_n\text{'}} \qquad := \{s \in R : s \notin \mathfrak{p} \quad \forall \mathfrak{p} \in D(f)\}$$

depends only on $D(f)$ and not on f. Fact: $S_{D(f)}^{-1} R \xleftarrow{\sim} R_f$. The restriction maps are localisations:

$$\begin{aligned} D(g) \subset D(f) &\implies S_{D(f)} \subset S_{D(g)} \\ &\implies S_{D(f)}^{-1} R \xrightarrow{\rho} S_{D(g)}^{-1} R \end{aligned}$$



$\boxed{\text{II}}$ Check that $\mathcal{O}$ 'satisfies sheaf conditions' on the basis $\{D(f)\}_{f\in R}$, i.e., $\mathcal{O}$ is a 'sheaf on a basis'. The sheaf conditions can be reformulated algebraically in this case as follows. Let $D(f) = \cup_{i\in I} D(f_i)$ be an open cover. Denote localisation maps:

$$\rho_i : R_f \to R_{f_i}; \quad \rho_{ij} : R_{f_i} \to R_{f_i f_j}$$

Then $\mathcal{O}$ being a sheaf on $\{D(f)\}$ is equivalent to the following sequences being exact:

$$0 \longrightarrow R_f \xrightarrow{\alpha} \prod_{i\in I} R_{f_i} \xrightarrow{\beta} \prod_{i,j\in I} R_{f_i f_j}$$

where $\alpha(a) = \rho_i(a)$ and $\beta((a_i))_{i,j} = (\rho_{ij}(a_i) - \rho_{ji}(a_j))$. That means:

- α is injective (locality); 'sections agree locally implies agree globally'.
- $\ker\beta = \operatorname{im}\alpha$ (gluing); 'sections agreeing on overlaps can be glued'.

$\boxed{\text{Locality}}$ Want: $\alpha, \beta \in R_f$ and $\alpha|_{R_{f_i}} = \beta|_{R_{f_i}}$ for all i imply $\alpha = \beta$. By replacing X, R with $D(f), R_f$ we can assume $f = 1$, $R_f = R$, $D(f) = X$. $\alpha - \beta = 0 \in R_{f_i}$ implies by definition that $f_i^{N_i}(\alpha - \beta) = 0$ for some $N_i \in \mathbb{N}$ where N_i depends on i, but $\operatorname{Spec} R$ is quasi-compact so we can pick a finite subcover by $D(f_i)$ and let $N := \max_i N_i$. We get:

$$\begin{aligned} f_i^N(\alpha-\beta) = 0 \text{ for all } i &\implies \underbrace{(f_i^N)_i}_{R}(\alpha-\beta) = 0 \\ &\implies 1(\alpha-\beta) = 0 \\ &\implies \alpha = \beta \end{aligned}$$

because $\operatorname{Spec} R = \bigcup_i D(f_i) = \bigcup_i (f_i^N)$. $\boxed{\text{Gluing}}$: we have $s_i \in R_{f_i}$ such that $s_i|_{R_{f_i f_j}} = s_j = |_{R_{f_i f_j}}$. Want: $s \in R_f = R$ (assume $f = 1$) such that $s|_{R_{f_i}} = s_i$ for all i. Can assume $X = \operatorname{Spec} R = \bigcup_{i=1}^n D(f_i)$ finite cover, $s_i = g_i/f_i^{n_i}$ and assume $n_i = 1$ (because $D(f_i) = D(f_i^{n_i})$). Know: $s_i = s_j$ in $R_{f_i f_j}$ implies $(f_i f_j)^N(f_j g_i - f_i g_j)$ (pick some big N that works for all (i,j) - finitely many). Rewrite:

$$\underbrace{(f_j^{N+1})}_{b_j}\underbrace{(f_i^N g_i)}_{a_i} - \underbrace{(f_i^{N+1})}_{b_i}\underbrace{(f_j^N g_i)}_{a_j} = 0$$

Notice: $s_i = a_i/b_i$ and $D(f_i) = D(b_i)$ so we can assume $N = 0$ and $\boxed{f_j g_i = g_j f_i}$. We have:

$$\begin{aligned} \operatorname{Spec} R = \bigcup_{i=1}^{n} D(f_i) &\implies 1 = \sum r_i f_i \\ &\implies 1 g_j = \sum r_i f_i g_j \\ &= \sum r_i f_j g_i \quad \text{boxed equation} \\ &= f_j \sum r_i g_i \end{aligned}$$

And so: $s_j = g_j/f_j = \sum r_i g_i/1 \in R_f$ for all j implies we have globalised $s_j \in R_{f_j}$ to $s = \underbrace{\sum r_i g_i}_{\text{global section}} \in R = \mathcal{O}_X(X)$. □

Lecture 6

Last time:

We defined the structure sheaf $\mathcal{O}_{\operatorname{Spec} R}$ as a sheaf on the basis $\{D(f)\}_{f\in R}$ with the Zariski topology on $\operatorname{Spec} R$ such that $\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R_f$.

Left to do:

- extend $\mathcal{O}_{\operatorname{Spec} R}$ to all $U \subset \operatorname{Spec} R$ open
- compute stalks $\mathcal{O}_{\operatorname{Spec} R,x}$ for all $x \in \operatorname{Spec} R$.

$\boxed{\text{III}}$ Define $\mathcal{O}_{\operatorname{Spec} R}$ to the unique sheaf extending $\mathcal{O}$ from the basis $\{D(f)\}_{f\in R}$ (general construction: see Alex Ritter's notes)

$$\begin{aligned}\mathcal{O}_{\operatorname{Spec} R}(U) &:= \lim_{D(f)\subset U} \mathcal{O}(D(f)) \\ &= \lim_{D(f)\subset U} R_f \\ &:= \left\{ (s_f) \subset \bigcap_{D(f)\subset U} R_f : s_f|_{D(g)} = s_g \quad \forall D(g) \subset D(f) \subset U \right\}\end{aligned}$$

'compatible families of local sections on basic open sets $D(f) \subset U$'

Intuition: 'lim genaralise $\bigcap$, colim generalise $\bigcup$ to the situation where $R_f \to R_g$ may not be injective'.

$\boxed{\text{IV}}$ We can now compute the stalks:

$$\begin{aligned}\mathcal{O}_{\operatorname{Spec} R,x} &= \operatorname{colim}_{U\ni x} \mathcal{O}_{\operatorname{Spec} R}(U) \\ &= \operatorname{colim}_{D(f)\ni x} \mathcal{O}_{\operatorname{Spec} R}(D(f)) \\ &= \operatorname{colim}_{f\notin \mathfrak{p}_x} \\ &= R_{\mathfrak{p}_x}\end{aligned}$$

Remark 4.3. For all $U \subset \operatorname{Spec} R$ open, $\mathcal{O}_{\operatorname{Spec} R}(U)$ is an R-algebra.

$$[a] : R_f \xrightarrow{\sim} R_f \text{ on } D(f)$$

induces R-module structure on $\mathcal{O}_{\operatorname{Spec} R}(U)$ for all open U

map of sheaves $[a] : \mathcal{O}_{\operatorname{Spec} R} \to \mathcal{O}_{\operatorname{Spec} R}$

4.2. **Affine schemes.**

We need ringed spaces because $\mathcal{O}_x$ is not given by k-valued functions for some k.

Definition 4.4. A *ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings of X.

N.B. A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ where $f : X \to Y$ is a continuous map of topological spaces and $\boxed{f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X}$ is a map of sheaves of rings on Y, or equivalently, $f^\# : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. That is, for all $U \subset Y$ open, we have *extra data* of a ring homomorphism $f^\#(U) : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$ such that for all $V \subset U$ the following square commutes:

$$\begin{tikzcd} \mathcal{O}_Y(U) \arrow[r] \arrow[d] & \mathcal{O}_X(f^{-1}U) \arrow[d] \\ \mathcal{O}_Y(V) \arrow[r] & \mathcal{O}_X(f^{-1}V) \end{tikzcd}$$

Remark 4.5.
1. $f^\#$ generalises pullback of regular functions on k-varieties:
$$(h : \underbrace{U}_{\subset Y} \to k) \mapsto (h \circ f : f^{-1}U \to k)$$
The difference is that unlike for varieties we have to record $f^\#$ as extra data because we don't have these pullback maps for free.
2. For all $x \in X$, let $y := f(x)$. then there's an induced map
$$f^\#_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$
that sends (s, V) with $y \in V \subset Y$ open to $(f^\# s, f^{-1}V)$ where $x \in f^{-1}V \subset X$ open. The map respects $\sim$ because $f^\#$ commutes with ρ_{UV}.

Definition 4.6. A *locally-ringed space* is a ringed apace $(X, \mathcal{O}_X)$ such that for all $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring. A morphism of locally-ringed space is a morphism of ringed space such that for all $x \in X$ and $y := f(x)$, the induced map $f^\#_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a local ring homomorphism, i.e., $f^\#_x(\mathfrak{m}_y) \subset \mathfrak{m}_x$, or equivaletly, $(f^\#_x)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_y$

Remark 4.7. For k-varieties, this condition was automatic: $\mathfrak{m}_y = \{f \in \mathcal{O}_{Y,y} : f(y) = 0\}$ implies $f^\# \mathfrak{m}_y \subset \mathfrak{m}_x$.

Remark 4.8. $f^\#$ local induces field extension on residue fields
$$\kappa(f(x)) := \mathcal{O}_{Y,f(x)}/\mathfrak{m}_{f(x)} \hookrightarrow \mathcal{O}_{X,x} frm_x =: \kappa(x)$$

Main example: $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a locally ringed space.

Proposition 4.9. *A ring homomorphism $\varphi : R \to S$ induces a map of locally-ringed space*
$$\operatorname{Spec} \varphi = (\varphi^*, \varphi^\#) : \operatorname{Spec} S \to \operatorname{Spec} R$$
that satisfies

1. *On distinguished open set $D(f)$ with $f \in R$*
$$R_f = \mathcal{O}_{\operatorname{Spec} R}(D(f)) \xrightarrow{\varphi^\#(D(f))} \mathcal{O}_{\operatorname{Spec} R}(D(\varphi(f))) = S_{\varphi(f)}; a/f^n \mapsto \varphi(a)/\varphi(f)^n$$
is the localisation of φ at f.
2. *On stalks, for all $\mathfrak{p} \in \operatorname{Spec} S$, the map $\varphi^\# : R_{\varphi^{-1}(\mathfrak{p})} \to S_\mathfrak{p}$ is the localisation of the φ.*

Proof sketch.
- Define $\varphi^\#$ on $D(f)$ as in (1)
- Check compatibility with ρ_{UV}
- Compute $\varphi^\#$ on stalks as in (2)

□

Definition 4.10. An *affine scheme* is a locally ringed space isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$. The category of affine schemes **AffSch** is a full subcategory of locally-ringed spaces. We have a functor:
$$\operatorname{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{AffSch}$$
We also have the *global section functor*:
$$\Gamma : \mathbf{AffSch}^{\mathrm{op}} \to \mathbf{Ring}$$
$$(X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X) =: \Gamma(X, \mathcal{O}_X)$$
$$(f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)) \mapsto (f^\#(Y) : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X))$$

Theorem 4.11. *The functor* Spec : $\mathbf{Ring}^{\mathrm{op}} \to \mathbf{AffSch}$ *is an equivalence of categories with inverse functor* Γ. *In particular,* $f : \operatorname{Spec} S \to \operatorname{Spec} R$ *is an isomorphism of ring spaces if and only if* $f^{\#} : R \to S$ *is an isomorphism of rings.*

Proof. We need to show for $X = \operatorname{Spec} S$, $Y = \operatorname{Spec} R$ and $f : X \to Y \in \mathbf{AffSch}$, that $\operatorname{Spec}(\Gamma(f)) = f$. Let $\varphi : \Gamma(f) = f^{\#}(Y) : R \to S$. Let $x \in X$ correspond to $\mathfrak{q} \in \operatorname{Spec} S$ and $f(x)$ correspond to $\mathfrak{p} \in \operatorname{Spec} R$. We want $f = \operatorname{Spec} \varphi$ and $f^{\#} = (\operatorname{Spec} \mathfrak{q})^{\#}$. Want $\operatorname{Spec} \varphi = f$ as a map of topological spaces, i.e., that $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$. We have:

$$\begin{tikzcd}
R \arrow[r, "\varphi"] \arrow[d] & S \arrow[d] \\
R_{\mathfrak{p}} \arrow[r, "f^{\#}"] & S_{\mathfrak{q}}
\end{tikzcd}$$

commutes. Hence $\varphi(R \setminus \mathfrak{p}) \subset S \setminus \mathfrak{q}$, so $\varphi^{-1}(\mathfrak{q}) \subset \mathfrak{p}$, however $f^{\#}_x$ is local, so $\varphi^{-1}(\mathfrak{q}) \supset \mathfrak{p}$, therefore $\varphi^{-1} = \mathfrak{p}$. Also: for all x, the stalk map $f^{\#}_x$ must be the localisation of φ, i.e., $(\operatorname{Spec} \varphi)^{\#}_x$, because the universal property of localisation gives us the commuting square

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & S \arrow[d] \\
R_{\mathfrak{p}} \arrow[r, dashed, "\exists!"] & S_{\mathfrak{q}}
\end{tikzcd}$$

Similarly $f^{\#}(D(h)) : R_h \to S_{\varphi(h)}$ is the localisation of φ at $h \in R$. Hence maps of sheaves $\operatorname{Spec} \varphi^{\#}$ and $f^{\#}$ coincide. □

Lecture 7

5. Schemes

Definition 5.1. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine scheme: $X = \bigcup_{i \in I} U_i$ open cover such that for all i there is a ring R_i such that $(U_i, \mathcal{O}_X|_{U_i}) \simeq (\operatorname{Spec} R_i, \mathcal{O}_{\operatorname{Spec} R_i})$. For each $x \in X$, the stalk $\mathcal{O}_{X,x}$ is the *local ring* at (of) x. If $x \in U = \operatorname{Spec} R \subset X$ open, then $\mathcal{O}_{X,x} = \mathcal{O}_{U,x} = R_\mathfrak{p}$ where $\mathfrak{p} = \mathfrak{p}_x$. The *residue field at x* is defined as $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}\mathcal{O}_{X,x} = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. A *morphism* (map of schemes) is a map $(f, f^\#)$ of locally ring spaces so that the category of affine schemes **AffSch** is a full subcategory of the category of schemes **Sch**. Let $\mathbb{F}$ be any field and X a scheme. The set is called

$$X(\mathbb{F}) := \{\operatorname{Spec} \mathbb{F} \to X\}$$

the set of *$\mathbb{F}$-points of X*, and more generally they are known as *schematic points.*

If $x \in X$, then for any open affine subset $U \subset X$ containing x, we have the inclusions

$$\operatorname{Spec} \underbrace{\kappa(x)}_{R_\mathfrak{p}/\mathfrak{p}R} \to \underbrace{U}_{\operatorname{Spec} R} \hookrightarrow X$$

Theorem 5.2. *Let X be a scheme, and R be a ring. Then*

$$\operatorname{Maps}_{\mathbf{Sch}}(X, \operatorname{Spec} R) \simeq \operatorname{Maps}_{\mathbf{Ring}}(R, \mathcal{O}_X(X))$$

So giving a map $X \to \operatorname{Spec} R$ is the same as giving an R-algebra structure on $\mathcal{O}_X$

Proof sketch. WLOG $X = \operatorname{Spec} S$ (proved last time). For a general X we'll define a map $\leftarrow$: given $\varphi : R \to \mathcal{O}_X(X)$, for all $x \in X$

$$\begin{tikzcd}
R \arrow[r] & \mathcal{O}_X(X) \arrow[r] & \mathcal{O}_{X,x} \\
\psi_x^{-1}(\mathfrak{m}_x) \arrow[u, hook] \arrow[rr, dashed] & & \mathfrak{m}_x \arrow[u, hook]
\end{tikzcd}$$

Define $g : X \to \operatorname{Spec} R; x \mapsto \psi_x^{-1}(\mathfrak{m}_x)$. The map g is continuous because we can check that $g^{-1}(D(f)) = D(\varphi(f))$. And then

$$\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R_f \xrightarrow{\varphi f} \mathcal{O}_X(X)_{\varphi(f)} \xrightarrow{*} \mathcal{O}_X(D(\varphi(f))) = \mathcal{O}_X(g^{-1}D(f)) = g_*(g^{-1}D(f))$$

where $*$ factors through the localisation at $\varphi(f)$ because $\varphi(f)$ is invertible in $\mathcal{O}_X(D(\varphi(f)))$. □

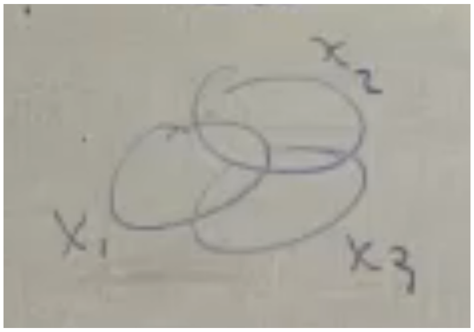
Corollary 5.3 ('$\mathcal{O}_X$ encodes functions on X')**.**

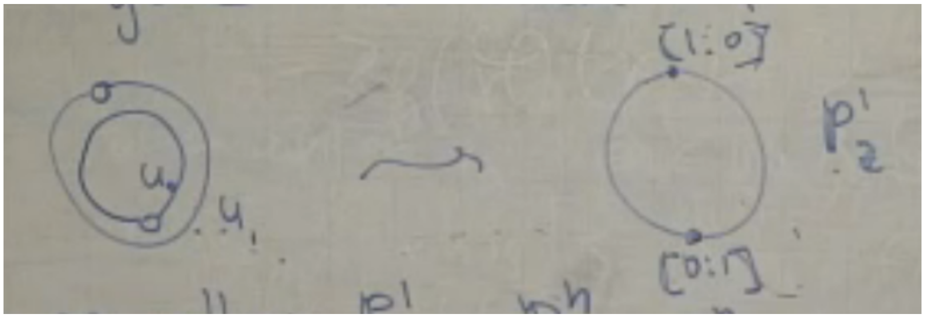
$$\operatorname{Maps}(X, \mathbb{A}^1 \simeq \mathcal{O}_X(X))$$

since $\mathbb{Z}[x] \to \mathcal{O}_x(X)$ is determined uniquely by the image of x.

Example 5.4 (Open subschemes)**.** Let $(X, \mathcal{O}_X)$ be a scheme and $U \subset X$ and open subset. Then $(U, \mathcal{O}_X|_U)$ is also a scheme. This is because for the distinguished open set $U(x) \subset U$ we have that $(U(x), \mathcal{O}_U|_{U(x)})$ is an affine scheme.

Example 5.5 (Non-affine scheme)**.** $U := \mathbb{A}^2 \setminus \{(0,0)\} \subset \mathbb{A}^2 = \operatorname{Spec} \mathbb{Z}[x, y]$





Exercise 5.6. $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2) \to \mathcal{O}_U(U)$ is an isomorphism but $U \subset \mathbb{A}^2$ is not because

$$Z(x,y) \begin{cases} = \emptyset & \text{in } U \\ \neq \emptyset & \text{in } \mathbb{A}^2 \end{cases}$$

so U cannot be affine.

Gluing: how to get non-affine schemes. Idea: let X_i be 'schemes that agree on intersections', i.e., specify $\mathcal{O}_{X_i}|_{X_i \cap X_j} \simeq \mathcal{O}_{X_j}|_{X_i \cap X_j}$

Example 5.7 (Projective line)**.** Let

$$\begin{aligned} U_0 &:= \operatorname{Spec} \mathbb{Z}[u] = \mathbb{A}^1 & U_1 &:= \operatorname{Spec} \mathbb{Z}[u^{-1}] = \mathbb{A}^1 \\ U_{01} &:= D(u) = \operatorname{Spec} \mathbb{Z}[u, u^{-1}] & D_{10} &:= D(u^{-1}) = \operatorname{Spec} \mathbb{Z}[u^{\pm 1}] \end{aligned}$$

$\mathbb{P}^1$: glue U_0 and U_1 along $U_{01} \simeq U_{10}$ for $a \neq 0$, $[1 : a] = [1/a : 1]$ (coordinates in different charts). More generally: $\mathbb{P}^1_R; \mathbb{P}^n_{\mathbb{Z}}, \mathbb{P}^n_R$.

Remark 5.8. $\mathbb{P}^n_R$ can be differently defined using 'Proj' (Hartshorne, Vakil,...).

5.1. **Integral scheme.**

Definition 5.9. A scheme $(X, \mathcal{O}_X)$ is reduced if each $\mathcal{O}_{X,x}$ is reduced (no nilpotents).

Exercise 5.10. X is reduced if and only if $\mathcal{O}_X(U)$ is reduced for all affine open spaces. $\operatorname{Spec} R$ is reduced if and only if R is reduced.

Associated reduces schemes: $\operatorname{Spec} R_{\text{red}} \underbrace{\hookrightarrow}_{\text{'closed immersion'}} \operatorname{Spec} R$ where $R_{\text{red}} := R/\operatorname{Nil} R$.
These are the same topological spaces but with different structure sheaves.

Example 5.11. If $R = k[t]/t^n$, then $\operatorname{Spec} R_{\text{red}} = \operatorname{Spec} k \hookrightarrow \operatorname{Spec} R$. Claim: For any scheme X, one can glue $X_{\text{red}} \hookrightarrow X$, and it is universal: For any Y reduced scheme

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, dashed, "\exists!"] & X \\
X_{\text{red}} \arrow[ur] &
\end{tikzcd}$$

Definition 5.12. A scheme is *integral* if it is reduced and irreducible.

Proposition 5.13. *$(X, \mathcal{O}_X)$ is integral if and only if for each (affine) open set $U \subset X$, $\mathcal{O}_X(U)$ is an integral domain.*

Proof for $\operatorname{Spec} R$.

$$\begin{aligned}
\operatorname{Spec} R \text{ integral} &\iff \operatorname{Nil} R = (0) \text{ and is prime} \\
&\iff (0) \text{ is prime} \\
&\iff R \text{ is an integral domain}
\end{aligned}$$

□

Structure sheaf of an integral scheme: 'sections of $\mathcal{O}_X$ can be thought of as certain rational functions'.

Definition 5.14. Let X be an integral scheme, and let $\eta \in X$ be the generic point (which exists because X is irreducible). The *function field* of X is $\kappa(X) := \mathcal{O}_{X,\eta}$. It is a field because for any $\operatorname{Spec} R \subset X$ open:

$$\mathcal{O}_{X,\eta} = \mathcal{O}_{\operatorname{Spec} R,\eta} = R_{(0)} = \operatorname{Frac} \underbrace{R}_{\text{integral domain}}$$

Proposition 5.15. *Let X be integral, $U \subset X$ open, $\eta \in X$ the generic point.*

(1) $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} = \kappa(X)$ *is injective*
(2) *For any $V \subset U$ open, $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is injective*
(3) $\mathcal{O}_{X,x} \subset \kappa(X)$ *for all $x \in X$ and $U \ni x$ implies $\mathcal{O}_X(U) \subset \mathcal{O}_{X,x}$*
(4) $\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x} \subset \kappa(X)$ *If $X = \operatorname{Spec} R$, then*

$$\mathcal{O}_X(U) = \{f \in \kappa(X) : f = g/h \quad g, h \in R \text{ and } h(U) \neq 0\}$$

Example 5.16. $X = \mathbb{A}^n_k = \operatorname{Spec} k[t_1, \ldots, t_n]$ implies $\kappa(X) = k[t_1, \ldots, t_n]$.

LECTURE 8

Previously:

Proposition 5.17. *Let X be an integral scheme, $\eta \in X$ be the generic point and let $U \subset X$ be an open subset.*

1. *The map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} := \kappa(X)$ is injective.*
2. *For any $V \subset U$ open, $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is injective.*
3. *For all $x \in U$ we have and $\mathcal{O}_X(U) \subset \mathcal{O}_{X,x} \subset \kappa(X)$.*
4. *$\mathcal{O}_X(U) = \bigcap_{x \in X} \mathcal{O}_{X,x} \subset \kappa(X)$*
   If $X = \operatorname{Spec} R$, then

$$\mathcal{O}_X(U) = \{f \in \kappa(X) : \textit{for all } x \in U \textit{ there are } g, h \in R \textit{ such that } f = g/h \textit{ where } h(x) \neq 0\}.$$

Proof. (1) Let $f \in \mathcal{O}_X(U)$, assume $f(\eta) = 0$. Then for all affine open $V = \operatorname{Spec} S \subset U$, we have $\rho_{UV}(f) = 0$ because S is an integral domain, hence $S \hookrightarrow \operatorname{Frac} S = \kappa(X)$. Take an affine open cover $U = \bigcup_i V_i$ so that if $\rho_{UV_i}(f) = 0$ for all i, then $f = 0$ because $\mathcal{O}_X$ is a sheaf.

(2) The inclusion maps $\mathcal{O}_X(U) \hookrightarrow \kappa(X)$ are compatible with restriction maps:

$$\begin{tikzcd}
\mathcal{O}_X(U) \arrow[rr, "\rho_{UV}"] \arrow[dr, hook] & & \mathcal{O}_X(V) \arrow[dl, hook'] \\
& \kappa(X) &
\end{tikzcd}$$

so ρ_{UV} is injective.

(3) The canonical map

$$\mathcal{O}_{X,x} \to \mathcal{O}_{X,\eta}; [U, f] \mapsto [U, f]$$

is injective: Think of the stalk as $\mathcal{O}_{X,x} = \mathcal{O}_{V,x} = A_p \hookrightarrow \operatorname{Frac} A = \mathcal{O}_{X,\eta}$ where $x \in V$ and $V = \operatorname{Spec} A$ is an affine neighbourhood. For $U \ni x$, $\mathcal{O}_X(U) \hookrightarrow \mathcal{O}_{X,x} \hookrightarrow \kappa(X)$. By (3) $\mathcal{O}_X(U) \subset \bigcap \mathcal{O}_{X,x}$. Let $f \in \bigcap \mathcal{O}_{X,x} \subset \kappa(X)$. For all x there exists an open get $V(x) \subset U$ containing x with $f \in \mathcal{O}_X(V(x))$. Now $U = \bigcup_x V(x)$ so can glue $f \in \mathcal{O}_X(U)$ because $f|_{V(x) \cap V(x')}$ will agree as they coincide in $\kappa(X)$. The last formula follows because $X = \operatorname{Spec} R$ implies $\mathcal{O}_{X,x} = R_{\mathfrak{p}_x}$. □

6. FIBRE PRODUCTS AND ALL THAT JAZZ

Definition 6.1.
1. A morphism of schemes $f : X \to Y$ is called an *open immersion* if it is an isomorphism onto an open subscheme of Y, i.e., onto $(U, \mathcal{O}_Y|_U)$ for some open $U \subset Y$.
2. A morphism of schemes $g : X \to Y$ is called a *closed immersion* if $g^{\#}$ is a homeomorphism onto a closed subset of Y and $g^{\#} : \mathcal{O}_Y \to g_*\mathcal{O}_X$ is surjective. For example
$$\operatorname{Spec} k \not\hookrightarrow \operatorname{Spec} k[t]/t^n \hookrightarrow \operatorname{Spec} k[t]$$
3. A *closed subscheme* of Y is an equivalence class of closed immersions into Y: so $[X \not\hookrightarrow Y] \sim [X' \hookrightarrow Y']$ if and only if
$$\begin{tikzcd}
X' \arrow[rr, "\sim"] \arrow[dr] & & X \arrow[dl] \\
& Y &
\end{tikzcd}$$

Definition 6.2. Let S be a scheme. An *S-scheme* is a scheme X with a chosen map $X \to S$ called the *structure morphism*, in this case we call S the *base scheme*. A morphism of S-schemes is

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\
& S &
\end{tikzcd}$$

This gives us a the *category of S-schemes* $\mathbf{Sch}_S := \mathbf{Sch}_{\operatorname{Spec} S}$. For example $\mathbf{Sch} = \mathbf{Sch}_{\mathbb{Z}}$.

6.1. **Fibre products.** Motivation: Fibre products help us to

- Define the right notion of product in $\mathbf{Sch}_S$.
- For $X_1 \not\hookrightarrow Y$, $X_2 \not\hookrightarrow$ we can define '$X_1 \cap X_2$' as a scheme.
- For a morphism of schemes $f : X \to Y$ and $y \in Y$ we can define $f^{-1}(y)$ as a scheme.
- Obtain $\mathbb{P}^n_R$ from $\mathbb{P}^n_{\mathbb{Z}}$ and $\mathbb{Z} \hookrightarrow R$ (e.g. $R = \mathbb{C}$).

Definition 6.3. Let $f : X \to S$, and $g : Y \to S$. The *fibre product* is a scheme $X \times_S Y$ with the universal commutative square

$$\begin{tikzcd}
X \times_S Y \arrow[r, "p_Y"] \arrow[d, "p_X"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\
X \arrow[r] & S
\end{tikzcd}$$

Remark 6.4.
1. If $X \times_S Y$ exists, then it is unique (up to a unique isomorphism).
2. It makes sense in any category (may not exist!).
3. In **Sets**: $X \times_S Y \subset X \times Y$ is the subset $(x, y) \in X \times Y$ such that $f(x) = g(y) \in S$.

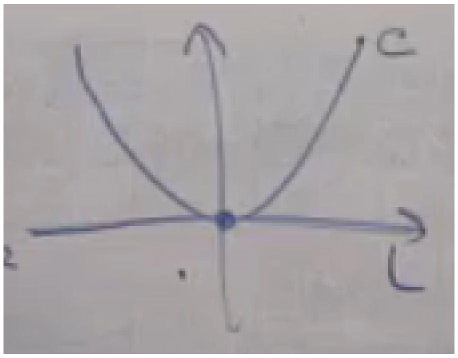
Theorem 6.5 (Hartshorne Theorem 3.3)**.** *Fibre products exist in* $\mathbf{Sch}_S$.

Remark 6.6. Often $S = \operatorname{Spec} Z$ gives $X \times Y \in \mathbf{Sch}$, but $S = \operatorname{Spec} k$ gives $X \times_k Y \in \mathbf{Sch}_k$. As a set, $X \times Y$ is the Cartesian product, but it has a different topology!

Example 6.7. $\mathbb{A}^n \simeq \mathbb{A}^1 \times \cdots \times \mathbb{A}^1$ - product in **Sch**, but does not have the product topology.

$\{xy = 1\} \subset \mathbb{A}^2$ is a closed subset in $\mathbb{A}^2$, but is not a Cartesian product.

Sketch Proof.
1. Affine case: let X, Y, S be schemes associated to the rings A, B, R and let X and Y be S-schemes. Then $\operatorname{Spec} A \times_R B$ does the job. We get $Z \to \operatorname{Spec} A \otimes_R B$ corresponds to $A \otimes_R B \to \Gamma(Z, \mathcal{O}_Z)$, which corresponds to R-module maps $A, B \to \Gamma(Z, \mathcal{O}_Z)$.
2. If $X \times_S Y$ exists and $U \subset X$ is open, then $U \times_S Y$ exists: take $p_X^{-1}(U)$ with the open subscheme structure.
3. If $X = \bigcup_i U_i$ and $U_i \times_S Y$ exists for all i, they can be glued into $X \times_S Y$: glue U_i's into X and glue maps to Y.
4. 1,2,3 imply that when Y, S is affine, that $X \times_S Y$ exists for all S. Symmetric in X and Y implies that $X \times_S Y$ exists when S affine.
5. Let $S = \bigcup_i S_i$ be an affine open cover. $X_i := f^{-1}(S_i)$, $Y_i := g^{-1}(S_i)$, so $X_i \times_{S_i} Y_i$ exits. Note $X_i \times_{S_i} Y_i = X_i \times_S Y$ (think about sets!).



(6) Glue $X_i \times_S Y$ again and you win!

$\square$

Example 6.8 (Base change)**.** This generalises the notion of changing the coefficients of equations.

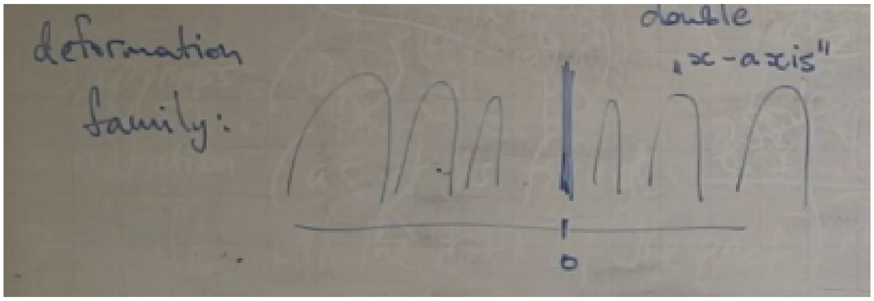
$$\mathbb{A}^n_R = \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} R$$
$$\mathbb{P}^n_R = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} R$$

Also works for $\mathbb{A}^n_X$, $\mathbb{P}^n_X$ for any scheme X. Actually: for all S-schemes T, X, we call $T_X := T \times_S$ the *base change of* T *to* X.

Example 6.9 (Intersections)**.** Let

$$C := \operatorname{Spec} \mathbb{C}[x, y]/(y - x^2) \subset \mathbb{A}^2$$
$$L := \operatorname{Spec} \mathbb{C}[x, y]/(y) \subset \mathbb{A}^2$$

then '$C \cap L$' $:= C \times_{\mathbb{A}^2} L = \operatorname{Spec} \mathbb{C}[x]/(x^2)$, so we have a double point! If we have $Z, Z' \not\hookrightarrow X$, then '$Z \cap Z'$' $:= Z \times_X Z'$ gives the 'correct' notion of intersection.



Lecture 9

6.2. Examples of fibre products. Last time:

(1) base change (scalar extension)
(2) intersections

Today:

(3) deformations

$$\begin{tikzcd}
\operatorname{Spec}\mathbb{C}[x,y]/(y^2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec}\mathbb{C}[x,y,t]/(y^2+tx) \arrow[d] \\
\underbrace{\operatorname{Spec}\mathbb{C}[t]/(t)}_{\text{Closed point of 0}} \arrow[r, hook] & \operatorname{Spec}\mathbb{C}[t]
\end{tikzcd}$$

(4) schematic fibres

For any scheme S, we think of a point as the inclusion $\operatorname{Spec} \underbrace{\kappa(p)}_{A_p/pA_p} \hookrightarrow \operatorname{Spec} A \subset S$.

Definition 6.10. For any $\varphi : X \to S$, the *scheme-theoretic* fibre of φ at $p \in S$ is X_p.

$$\begin{tikzcd}
X_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
\operatorname{Spec}\kappa(p) \arrow[r, hook] & S
\end{tikzcd}$$

Remark 6.11. X_p is a $k(p)$-scheme for all $p \in S$!

Example 6.12. Let k be algebraically closed, and consider $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ induced by $k[x] \to k[y]; x \mapsto y^2$. The fibre over 0 is:

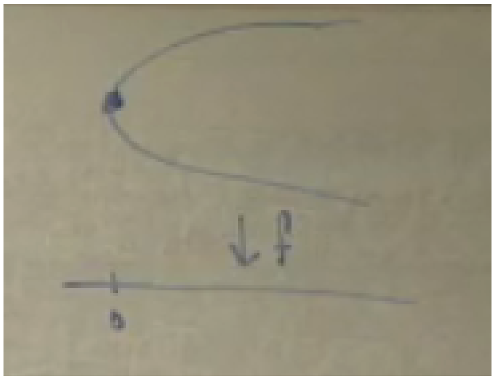
$$\operatorname{Spec}(k \otimes_{k[x]} k[y]) = \operatorname{Spec} k[y]/y^2$$

which corresponds to a double point.

(5) generic fibre of a map $\varphi : X \to S$

Definition 6.13. The *generic fibre* is the/a fibre over a generic point.

Moral: encodes 'general behaviour' (something that happens over a dense open subset)



Example 6.14.

$$\begin{tikzcd}
\overbrace{\operatorname{Spec}\mathbb{C}[x,y]/(y^2)}^{\text{family of conics}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec}\mathbb{C}[x,y,t]/(y^2+tx) \arrow[d] \\
\operatorname{Spec}\mathbb{C}[t]/(t) \arrow[r, hook] & \operatorname{Spec}\mathbb{C}[t]
\end{tikzcd}$$

6.3. **Separatedness.** A scheme is usually not Hausdorff as a topological space because of generic points. In topology there is a criterion about being Hausdorff:

$$X \text{ Hausdorff} \quad \Longleftrightarrow \quad \Delta_X \subset X \times X \text{ is closed}$$

The right hand side of the above equivalence is more suitable for geometric considerations.

Definition 6.15. Given an S-scheme $\varphi : X \to S$, the *diagonal map* is $\Delta_{X/S} : X \to X \times_S X$ induced by the universal property of the fibre product applied to φ.

Definition 6.16. A map $f : X \to S$ is *separated* if $\Delta_{X/S}$ is a closed immersion (or the S-scheme X given by f is separated).

Fact: it is enough to check that $\operatorname{im}(X) \subset X \times_S X$ is a closed subset.

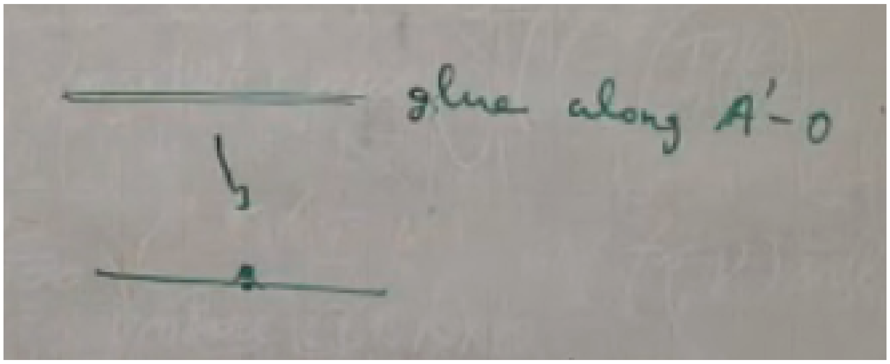
Example 6.17.
1. If $f : X \to S$ is an affine scheme, then f is separated because $A \otimes_B A \to A$ is always surjective.
2. $\mathbb{A}^n_S$, $\mathbb{P}^n_S$ are separated S-schemes for all affine schemes S.
3. The open fibre of a closed immersion is separated.
4. Compositions of separated maps are separated.
   Moral: ‘almost any scheme is separated except pathological ones’:
5. The bug-eyed line $\mathbb{A}^1 \cup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1$ is NOT separated.

6.4. **Varieties.**

Definition 6.18. A k-scheme X is of *finite type* is $X = \bigcup_{j=1}^m \operatorname{Spec} A_j$ for some finitely generated k-algebras A_j.

Exercise 6.19. Show this is equivalent to $\mathcal{O}_X(U)$ is a finitely generated k-algebra for all open $U \subset X$ and X is quasi-compact.

Definition 6.20. Let k be algebraically closed. A *variety* over k is a reduced, finite type, separated k-scheme (sometimes: also irreducible, or not separated).



C3.4 → C2.6 makes varieties *sober*.

Remark 6.21. All quasi-projective varieties from classical algebraic geometry are varieties in this sense, but not every variety is quasi projective by a counterexample $X \not\subset \mathbb{P}^{\geq}_{\urcorner}$ of Nagata.

7. Morphisms

7.1. **Properness.** In topology a space X is compact if and only if for all spaces Y, the projection map $X \times Y \to Y$ sends closed subsets to closed subset.

Definition 7.1.

(1) A map of schemes $f : X \to S$ is *closed* if $f(Z) \subset S$ is closed for any closed $Z \subset X$.

(2) (Relative compactness) A morphism $f : X \to S$ is *universally closed* if any base change of f is closed:

$$\begin{tikzcd}
X \times_S T \arrow[r] \arrow[d, "\text{this map is closed}"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\
T \arrow[r] & S
\end{tikzcd}$$

for all S-schemes T.

Non-example: closed but not universally closed. The map $\mathbb{A}^1_k \to \operatorname{Spec} k$ is not because $\mathbb{A}^1_k \times \mathbb{A}^1_k \to \mathbb{A}^1_k; (a,b) \mapsto b$ is not closed: $Z(xy-1) \mapsto k \setminus 0$.

Remark 7.2. We prefer properties that are preserved under base change (e.g. seprarated).

Definition 7.3. A map $f : X \to S$ is *proper* if it is universally closed, separated and finite type.

Definition 7.4. A map $f : X \to Y$ is *finite type* if there exists an open cover $Y = \bigcup_i V_i$ with $V_i = \operatorname{Spec} B_i$ such that for all i, $f^{-1}(V_i)$ has a finite open cover by $\operatorname{Spec} A_{i_j}$ where each A_{i_j} is a finitely generated B_i-algebra.

Figure 6. Local rings of nonsingular points of curves.

Lecture 10

Last time:
$f : X \to S$ is proper if and only if it is

- universally closed,
- separated and,
- finite type.

Remark 7.5. Properness is *stable under base change*: If $f : X \to Y$ is a proper morphism, then for all morphisms $Z \to Y$, $Z \times_Y X \to Z$ is proper.

Example 7.6. (1) $\mathbb{A}^n_R \to \operatorname{Spec} R$ is not proper.
(2) $\mathbb{P}^n_R \to \operatorname{Spec} R$ is proper. 'compactification of $\mathbb{A}^n_R$'

7.2. Valuative criterion.

Definition 7.7. A scheme X is *Noetherian* if $X = \bigcup_{i=1}^m \operatorname{Spec} A_i$, where the A_i are Noetherian rings (all ideals of A_i are finitely generated).

Theorem 7.8 (Valuation criterion of properness)**.** *Let $f : X \to Y$ be a map of schemes with X Noetherian. Then f is proper if and only if for any discrete valuation ring A with $K = \operatorname{Frac}(A)$, the following diagram commutes*

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists!"] & Y
\end{tikzcd}.$$

Reminder on dvr's:

- examples, $\mathbb{Z}_{(p)}$, $\mathbb{Z}_p$, $k[[T]]$.
- PID with one nonzero maximal ideal $\mathfrak{m}$.
- has a uniformiser: $\mathfrak{m} = (\varpi)$ and any ideal in A is (ϖ^k) where $k \in \mathbb{N}$.
- For all $a \in A$ we have $a = u\varpi^k$ for some $u \in A^\times$, $k \in \mathbb{N}$.
- For any $t \in K$ we have $t = u\varpi^k$ for some $u \in A^\times$, $k \in \mathbb{Z}$.

Applications:

1. $\mathbb{P}^n_{\mathbb{Z}}$ is proper (hence $\mathbb{P}^n_R \to R$ is proper for all R by base change). Pick a dvr A; $\operatorname{Frac}(A) = K$; ϖ is the uniformiser. We want $\mathbb{P}^n_{\mathbb{Z}}(K) \leftrightsquigarrow \mathbb{P}^n_{\mathbb{Z}}(A)$ is a bijection. A K-point of $\mathbb{P}^n_{\mathbb{Z}}$ is $[z_0 : \ldots : z_n]$ with $z_i \in K$ not all zero and $z_i = u_i\varpi^{k_i}$ where $k_i \in \mathbb{Z}$ for all i. Let $z'_i := \varpi^m z_i$ for some m such that $z'_i \in A$ for all i. Then $[z_0 : \cdots : z_n] = [z'_0 : \cdots : z'_n]$ is an A-point of $\mathbb{P}^n_{\mathbb{Z}}$.
2. $\mathbb{A}^n_k \to \operatorname{Spec} k$ is not proper: take $A = k[[T]]$, $K = k((T))$. Consider $\operatorname{Spec} K \to \mathbb{A}^n_k$ given by $(1/T, 1, \ldots, 1)$, it cannot be extended to an A-point because $1/T \notin A$!

(3) If $X \subset \mathbb{P}^n_{\mathbb{Z}}$ is closed, then $X \to \operatorname{Spec} \mathbb{Z}$ proper:
(4)

$$\begin{tikzcd}
X(A) \arrow[r, hook] & \mathbb{P}^n(A) \\
X(K) \arrow[r, hook] \arrow[u, "\therefore\text{bijection}"] & \mathbb{P}^n(K) \arrow[u, "\text{bijection}"']
\end{tikzcd}$$

Definition 7.9. A morphism $f : X \to Y$ is called

1. *projective*, if it can be factored as $X \not\hookrightarrow \mathbb{P}_{\mathbb{Y}}{}^m \xrightarrow{\text{pr}} Y$.
2. *quasi-projective*, if it can be factored as $X \underbrace{\hookrightarrow}_{\text{open immersion}} \mathbb{P}_{\mathbb{Y}}{}^m \xrightarrow{\text{pr}} Y$.

Fact: If Y is Noetherian, then f is proper, and most proper maps arise in this way. Fact: If X and Y are both Noetherian then f is quasi-projective if and only if f is of finite type and separated.

7.3. **Flatness.** Moral: flat maps 'encode continuously varying families'.

Definition 7.10. A say a map of schemes $f : X \to Y$ is *flat* if all the induced maps on stalks $\mathcal{O}_{Y,f(x)} \xrightarrow{f_x^*} \mathcal{O}_{X,x}$ are flat ring homomorphism, i.e., $\mathcal{O}_{X,x}$ is a flat module over $\mathcal{O}_{Y,f(x)}$ ($- \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$ sends injections to injections).

Basic facts:

- Free R-modules are flat R-modules for any ring R.
- A $\mathbb{Z}$-module is flat if and only if it is torsion-free. ($-\otimes\mathbb{Z}/n$ sends $[n] : \mathbb{Z} \hookrightarrow \mathbb{Z}$ to $[0] : \mathbb{Z}/n \to \mathbb{Z}/n$ - not injective).
- If M is a finitely generated R-module over a local ring R, then M is flat if and only if it is free.
- Localisations of flat modules are flat.

Exercise 7.11. $\varphi : A \to B$ is a flat ring homomorphism if and only if $\varphi^{\#} : \operatorname{Spec} B \to \operatorname{Spec} A$ is flat.

Example 7.12. (1) Open immersions of flat morphisms are flat; closed immersions are not flat.
(2) $\operatorname{Spec} k[x]/x^2 \to \operatorname{Spec} k$ is not flat. Intuition: $f : X \to Y$ being flat means that 'fibres vary in a controlled way' (weaker property than requiring all fibres to be isomorphic, but it allows to 'control' the differences between fibres).

Definition 7.13. Let X be a scheme and let $x \in X$ be a point. We define

$$\dim_x X := \sup\{r \in \mathbb{N} : \{x\} \subset Z_0 \subset \cdots \subset Z_r \subset U \text{ minimising over open } x \in U \subset X\}.$$

Example 7.14. $\dim_x \mathbb{A}^2 = 2$ for all x because {point} $\subset$ line $\subset$ plane.

Theorem 7.15. *Let X, Y be (locally) Noetherian schemes and $f : X \to Y$ a flat morphism with $x \in X$ and $y := f(x)$, then*

$$\dim_x f^{-1}(y) = \dim_x X - \dim_y Y.$$

Corollary 7.16 (Non-examinable)**.** *A blow-up in a closed point is not flat because it has one fibre that has different dimension than the others, but blow-ups are proper.*

Flatness is part of other interesting properties:

- If $f : X \to Y$ is smooth (Jacobians don't vanish), then f is flat.
- If $f : X \to Y$ is étale (smooth and relative dimension 0) if and only if f is flat and unramified.

Defining étale morphisms leads to étale cohomology which allowed to prove the Weil conjectures!

LECTURE 11

8. SHEAVES OF MODULES

8.1. $\mathscr{O}_X$-modules.

Definition 8.1. Let $(X, \mathscr{O}_X)$ be a ringed space. A *sheaf of $\mathscr{O}_X$-modules* is a sheaf $\mathscr{F}$ of abelian groups such that for all open sets $U \subset X$ there is a multiplication

$$\mathscr{O}_X(U) \times \mathscr{F}(U) \to \mathscr{F}$$

compatible with restriction maps. A *sheaf of $\mathscr{O}_x$-algebras* is the same as above but replace the category **Grp** with **Ring**.

Fact: They form an abelian category $\mathscr{O}_x$-**Mod**, so the following symbols are defined:

$$\text{ker; im; coker; } \bigoplus; \prod; \subset; \bigotimes; \text{ Hom}$$

N.B. For any sheaves $\mathscr{F}, \mathscr{G}$ we have $\mathscr{F} \otimes \mathscr{G}$ is the sheafification of $U \mapsto \mathscr{F} \otimes \mathscr{G}(U)$. Further $\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is defined as the sheaf $U \mapsto \operatorname{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U)$.

Remark 8.2. If $\mathscr{F}$ is an $\mathscr{O}_X$-module, then $\mathscr{F}_x$ is an $\mathscr{O}_{X,x}$-module and $\mathscr{F} \to \mathscr{F}'$ induces a map of $\mathscr{O}_{X,x}$-modules $\mathscr{F}_x \to \mathscr{F}'_x$.

Example 8.3. Let $X = \mathbb{P}^n_{\mathbb{C}}$ be a variety. We define a structure sheaf $\mathscr{O}_{\mathbb{P}^n}(d)$ as

$$U \mapsto \left\{ \frac{P(x_0, \ldots, x_n)}{Q(x_0, \ldots, x_n)} \text{ homogeneous of degree } d \text{ regular on } U \right\}.$$

Then $\mathscr{O}_{\mathbb{P}^n}(d)(\mathbb{P}^n)$ is the set of homogeneous polynomials of degree d in $x_0, \ldots, x_n$ and there is a multiplication $\mathscr{O}_{\mathbb{P}^n}(d)(U) \times \mathscr{O}_{\mathbb{P}^n}(d)(U) \to \mathscr{O}_{\mathbb{P}^n}(d)(U)$.

Moving between spaces:
Let $f : X \to Y$ be a morphism of ringed spaces:

- let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. Then
$$f_*\mathscr{F} \text{ is an } f_*\mathscr{O}_X\text{-module}$$
$$f_*\mathscr{F} \text{ is an } \mathscr{O}_Y\text{-module via } f^\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$$
- Let $\mathscr{G}$ be a sheaf of $\mathscr{O}_Y$-modules. Then
$$f^{-1}\mathscr{G} \text{ is an } f^{-1}\mathscr{O}_Y\text{-module}$$
$$f^*\mathscr{G}' := f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \text{ is an } \mathscr{O}_X\text{-module}$$

Claim: (f^*, f_*) are adjoint functors for modules over ringed spaces:

$$\operatorname{Hom}_{\mathscr{O}_X}(f^*\mathscr{G}, \mathscr{F}) \simeq \operatorname{Hom}_{\mathscr{O}_Y}(\mathscr{G}, f_*\mathscr{F})$$

for $\mathscr{F}, \mathscr{G}$ as above.

Fun-fact: Let $f : X \to Y$ be a flat morphism of schemes, then $f^* : \mathscr{O}_Y$-**Mod** $\to$ $\mathscr{O}_X$-**Mod** is an exact functor because $f^{-1}(-)$ does and so does $- \otimes_{f^{-1}\mathscr{O}_X} \mathscr{O}_X$.

8.2. (Quasi-)coherent sheaves.

Definition 8.4. Let R be a ring, M and R-module and $X = \operatorname{Spec} R$. Then the *sheaf associated to M* is

$$\widetilde{M} : D(f) \mapsto M_f$$

extended to a sheaf on $\operatorname{Spec} R$ in the same way as we defined $\mathscr{O}_X$.

In particular:

- $\widehat{M}(X) = M$
- $\widehat{M}_\mathfrak{p} = M_\mathfrak{p}$
- $\widehat{R} = \mathcal{O}_X$

and $\widetilde{M}$ is the sheafification of $U \mapsto M \otimes_R \mathcal{O}_X(U)$, note that this allows us to define $\widetilde{M}$ for any scheme $Y \to \operatorname{Spec} R$!

Definition 8.5. Let X be a scheme.

1. A *quasi-coherent* sheaf $\mathscr{F}$ on X is an $\mathcal{O}_X$-module such that there exists an affine open cover $X = \bigcup_{i \in O} U_i$ where $\varphi_i : \mathscr{F}|_{U_i} \cong \widetilde{M_i}$ for some $\mathcal{O}_X(U_i)$-modules M_i such that $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ for all $i, j, k \in I$.
2. We say $\mathscr{F}$ is *coherent* is all the M_i are finitely generated modules (works only when X is Noetherian).

Example 8.6. $\mathcal{O}_X^{\oplus n}$ is quasi-coherent (coherent when X is Noetherian).

Rethinking closed immersions:
$i : Z \not\hookrightarrow X$ homeomorphism onto a closed subset and $i^\# : \mathcal{O}_X \to i_* \mathcal{O}_Z$ is surjective. Denote $\mathcal{I}_{Z/X} := \ker i^\#$ the 'ideal defining Z in X'.

Lemma 8.7 (Easy)**.** (1) *$\mathcal{I}_{Z/X}$ is a* sheaf of ideals *on X, i.e., $\mathcal{I}_{Z/X}(U)$ is an ideal in $\mathcal{O}(U)$ for all $U \subset X$ open.*
(2) *$\mathcal{I}_{Z/X}$ is qcoh.; coh when X is Noeth.*
(3) *There is a bijection between qcoh sheaves of ideals and closed subschemes of X.*

Exercise 8.8 (Criterion)**.** An $\mathcal{O}_X$-module $\mathscr{F}$ is qcoh if and only if for all $U = \operatorname{Spec} R \subset X$ open, $\mathscr{F}|_U$ is a sheaf associated to an R-module M. If X is Noeth., then $\mathscr{F}$ is coh if all the M's are finitely generated.

Corollary 8.9. *If X is affine then*

$$\begin{aligned} \mathbf{QCoh}(X) &\xrightarrow{\sim} \mathcal{O}_X\text{-}\mathbf{Mod}; \\ \mathscr{F} &\mapsto \mathscr{F}(X) \\ \widetilde{M} &\leftarrow\!\shortmid M. \end{aligned}$$

Example 8.10 (Non-examples)**.** Not every $\mathcal{O}_X$-module is quasicoherent (although we like them a lot $\hat{}\smile\hat{})$

1. Let $X = \operatorname{Spec} k[x]_x = \{\mathfrak{m}, \eta\}$. Then $\mathscr{F}(X) : 0$, $\mathscr{F}(\eta) := \kappa(x)$ is a sheaf of $\mathcal{O}_X$-modules, but not quasicoherent, otherwise $\mathscr{F}(X) = 0$ would imply $\mathscr{F}(\eta) = 0$.
2. Let $X = \operatorname{Spec} k[t]$. Then
$$\mathscr{F}(U) := \begin{cases} \mathcal{O}_X(U) & \text{if } \{0\} \notin U \\ 0 & \text{if } \{0\} \in U \end{cases}$$
is not quasicoherent because $\mathscr{F} = 0$.
3. (Skyscraper sheaf) Let $X = \operatorname{Spec} k[x]$ and define
$$\mathscr{F}(U) := \begin{cases} k[x] & \text{if } 0 \in U \\ 0 & \text{else} \end{cases}$$
Then $\mathscr{F} \neq \widetilde{M}$ because $\widetilde{k[x]} = \mathcal{O}_X \ncong \mathscr{F}$.

8.3. **Properties of (quasi-)coherent sheaves.**

Proposition 8.11.

1. *Let X be Noetherian and let $f : \mathscr{F} \to \mathscr{G}$ be a morphism of (quasi-)coherent sheaves of $\mathcal{O}_X$-modules. Then* $\ker f$, $\operatorname{coker} f$ $\operatorname{im} f$ *are also (quasi-)coherent.*
2. *Let $f : X \to Y$ be a morphism of schemes and let $\mathscr{F}$ be a (quasi-)coherent $\mathcal{O}_Y$-module, then $f^*\mathscr{F}$ is also (quasi-)coherent (for coherent need X Noetherian because $f^*\mathcal{O}_Y = \mathcal{O}_X$).*
3. *Let $\mathscr{G}$ be a (quasi-)coherent sheaf on X, then it does* not *have to be the case that $f_*\mathscr{G}$ is (quasi-)coherent on Y (although $f_* : \mathbf{QCoh}(X) \to \mathbf{QCoh}(Y)$ when X is quasi-compact and separated).*

Example 8.12. Let $f : \mathbb{A}^1_k \to \operatorname{Spec} k$. Then $f_*\mathcal{O}_{\mathbb{A}^1_k} = \underbrace{k[x]}_{\text{not coherent!}} \in k\text{-}\mathbf{Mod}$ is not finitely generated.

Theorem 8.13 (Without proof)**.** *Let $f : X \to Y$ be a proper morphism with X, Y Noetheian. Then $f_*\mathbf{Coh}(X) \to \mathbf{Coh}(Y)$.*

Example 8.14. $Y \not\hookrightarrow X$ implies $i_*\mathcal{O}_Y$ is coherent. $X = \operatorname{Spec} R$ gives $i_*\mathcal{O}_Y = \widetilde{R/I}$ for $Y = \operatorname{Spec} R/I$.

Lecture 12

Last time:

If $f : X \to Y$ is a morphism of schemes, and $\mathscr{G}$ is quasicoherent on X, then it does *not* have to be the case that $f_*\mathscr{G}$ is quasicoherent on Y (but it is true when X is quasicompact and separable).

Example 8.15. Let $f : \coprod_{n\in\mathbb{N}} \mathbb{A}^1 \to \mathbb{A}^1$, $\mathscr{G} = \widetilde{\prod k[t]}$ on $\coprod \mathbb{A}^1$ if $f_*\mathscr{G} \in \mathbf{QCoh}(\mathbb{A}^1)$, then $f_*\mathscr{G}(D(t))$, however $(1/t^n)_{n\in\mathbb{N}} \notin \widetilde{\prod k[t]}(D(t)) = (\prod k[t])_t \neq \prod k[t]_t$.

Theorem 8.16 (Gabriel-Rosenberg Theorem, Non-examinable)**.** *Let X be a quasicompact and seprarable (e.g. X is a variety), then the abelian category* $\mathbf{QCoh}(X)$ *determines X up to an isomorphism!*

8.4. Vector bundles.

Definition 8.17. A sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ is a *vector bundle* if it is locally-free, that is, for all $x \in X$ there exists an open $U(x) \subset X$ such that $\mathscr{F}|_{U(x)} \simeq \mathcal{O}_X^n$ where $n \in \mathbb{N}$ is locally-constant; in the case $n = 1$ we say $\mathscr{F}$ is a *line bundle.*

N.B. It is not enough to ask if $\mathscr{F}_x \simeq \mathcal{O}_{X,x}$ for all x, however if $\mathscr{F}$ is coherent, then it is enough.

Construction: $\mathscr{F}$ can be encoded by the data $X = \bigcup_i U_i$ and

$$\begin{tikzcd}
\mathscr{F}|_{U_ij} \arrow[r, "\varphi_i", "\sim"'] \arrow[d, equal] & \mathcal{O}_{U_{ij}}^{n_i} \arrow[d, dashed, "\alpha_{ij}"] \\
\mathscr{F}|_{U_{ji}} \arrow[r, "\sim", "\varphi_j"'] & \mathcal{O}_{U_{ji}^{n_j}} = \mathcal{O}_{U_{ij}}^{n_j}
\end{tikzcd}$$

where the α_{ij} are transition maps that satisfy $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ on U_{ijk}.

Big picture:

$$\underbrace{\mathbf{Vect}(X)}_{\text{the nicest objects}} \subset \mathbf{Coh}(X) \subset \mathbf{QCoh}(X) \subset \mathcal{O}_X\text{-}\mathbf{Mod}$$

Note that $\mathbf{Vect}(X)$ is *not* an abelian category and ker and coker wouldn't be vector bundles.

Example 8.18.
1. $X = \operatorname{Spec} R$. Let $\mathscr{F}$ be a vector bundle, then $\mathscr{F} = \widetilde{M}$ where M is a finitely generated projective[1]
2. $X = \mathbb{P}^n$. Let $X = \bigcup_{i=0}^n A_i$ where $A_i = \operatorname{Spec} \mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}] \simeq A^n$
    - $\mathcal{O}(1)$: line bundle with $a_{ij} := \left(\frac{x_i}{x_j}\right)$ as the transition maps (multiply with this).
    - $\mathcal{O}(d)$ with $d \in \mathbb{Z}$ and transition maps $\alpha_{ij} := \left(\frac{x_i}{x_j}\right)^d$. Also: $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$ and $\mathcal{O}(-d) := \operatorname{Hom}(\mathcal{O}(d), \mathcal{O})$ for $d \geq 0$.

Exercise 8.19. Let $\mathbb{Z}[x_0, \ldots, x_n]_d$ denote d-homogeneous polynomials. Then

$$\Gamma(\mathbb{P}^n, \mathcal{O}(d)) = \begin{cases} \mathbb{Z}[x_0, \ldots, x_n]_d & d \geq 0 \\ 0 & \text{else} \end{cases}$$

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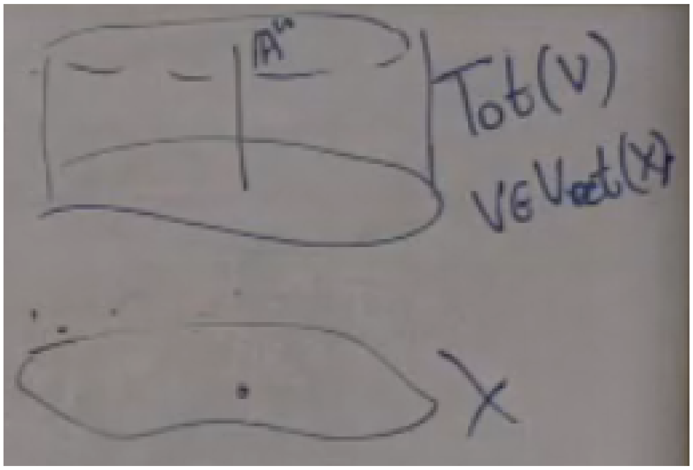
[1] Note this property is equivalent to being flat and equivalently locally-free when X is Noetherian.



Lemma 8.20. *A morphism of schemes $f : X \to Y$ induces a functor $f^* : \mathbf{Vect}(Y) \to \mathbf{Vect}(X)$.*

Sketch Proof.

(1) $f^*\mathcal{O}_Y = \mathcal{O}_X$.
(2) f^* commutes with the coproduct in $\mathcal{O}_X$-**Mod**.
(3) Can check locally.

□

Theorem 8.21. *Let $f : X \to Y$ be a finite flat morphism of schemes, and let $f_* : \mathbf{Vect}(X) \to \mathbf{Vect}(Y)$ be the induces functor. Then for affines $f_*(\widetilde{M}) = \widetilde{M}$ where $\widetilde{M}$ is considered as a module over $\mathcal{O}_Y(Y)$, and that's when scalar restriction preserves finitely generated projective (flat) modules.*

Remark 8.22. In general, f_* does not preserve **Vect**, e.g. when f is a closed immersion.

8.5. **Why vector bundles are called so?** Sketchy construction:

- Let $\mathcal{E} \in \mathbf{Vect}(X)$ be a locally-free $\mathcal{O}_X$-module of rank n.
- Define $\mathcal{E}^\vee := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ be locally-free of rank n.
- $\mathrm{Sym}\,\mathcal{E}^\vee$: locally-free sheaf of $\mathcal{O}_X$-algebras generalising: $V = k^n \rightsquigarrow \mathrm{Sym}\,V = k[x_1, \ldots, x_n]$. Namely $\mathrm{Sym}\,\mathscr{F} := \bigoplus_{m\geq 0}^{\otimes m} /(s \otimes t - t \otimes s)_{s,\,t \text{ local sections}}$. $\underline{\mathrm{Spec}}_X \mathrm{Sym}\,\mathcal{E}^\vee = \mathrm{Tot}(\varepsilon)$ is called the *total space of* $\mathcal{E} \in \mathbf{Vect(X)}$, and comes equipped with an X-scheme structute $\pi : \mathrm{Tot}(\varepsilon) \to X$ such that $\pi^{-1}(x) \simeq \mathbb{A}^n_{\kappa(x)}$ for all $x \in X$, and locally $\mathrm{Tot}(\varepsilon) \simeq \mathbb{A}^n \times U \to U$. In particular $\boxed{\underline{\mathrm{Spec}}_X \mathrm{Sym}(\mathcal{O}_S^{\oplus n}) = \mathbb{A}^n_S}$. More precisely: let $\mathscr{A}$ be a sheaf of $\mathcal{O}_X$-algebras (quasicoherent as an $\mathcal{O}_X$-module)
- Define a set $\underline{\mathrm{Spec}\,\mathscr{A} \xrightarrow{\pi} X}$ with $\pi^{-1}(p) = \mathrm{Spec}(\mathscr{A} \otimes \kappa(p))$.
- For all $U \subset X$ open, there exists a bijection $\pi^{-1}(U) \simeq \mathrm{Spec}\,\mathscr{A}(U)$.
- Define a topology and ring of functions on $\underline{\mathrm{Spec}\,\mathscr{A}}$ to make π a scheme map.

Why $\mathcal{E}^\vee$? Sections of $\mathcal{E}$ correspond to sections

$$\begin{tikzcd}
X \arrow[r, bend right, "s"'] & \mathrm{Tot}(\mathcal{E}) \arrow[l, bend right, "\pi"']
\end{tikzcd}$$

Because for all affine open $U \subset X$

$$\begin{aligned}
\{\text{Sections of } \mathrm{Tot}(\mathcal{E}) \to X\} &:= \mathrm{Hom}_{\mathbf{Sch}_X}(U, \underline{\mathrm{Spec}}\,\mathrm{Sym}\,\mathcal{E}^\vee) \\
&\simeq \mathrm{Hom}_{\mathbf{Alg}_{\mathcal{O}_X}}(\mathrm{Sym}\,\mathcal{E}^\vee(U), \mathcal{O}_X(U)) \quad \text{by const. of } \underline{\mathrm{Spec}} \\
&= \mathrm{Hom}_{\mathbf{Mod}_{\mathcal{O}_X(U)}}(\mathcal{E}^\vee(U), \mathcal{O}_X(U)) \quad \text{by universality of Sym} \\
&= \mathcal{E}^{\vee\vee}(U) \\
&\simeq \mathcal{E}(U) \quad \leftarrow \text{sections of } \mathcal{E} \text{ as a sheaf.}
\end{aligned}$$

Definition 8.23. An $\mathcal{O}_X$-module $\mathscr{L}$ *is invertible* if $\exists \mathscr{F} \in \mathbf{QCoh}(X)$

$$\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{F} \simeq \mathcal{O}_X$$

They form a group with respect to $- \otimes_{\mathcal{O}_X} -$.

Theorem 8.24. *$\mathscr{L} \in \mathcal{O}_X$-$\mathbf{Mod}$ is invertible if and only if $\mathscr{L}$ is a line bundle.*

Proof. (1) If $\mathscr{L}$ is a line bundle, then $\mathscr{L}^\vee := \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{L}, \mathscr{O}_X)$. Note $\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{L}^\vee \xrightarrow{\sim} \mathcal{O}_X$.

(2) If $\mathscr{L}$ is invertible, then locally on affine spaces $M \otimes_R N \xrightarrow{\sim} R$ (some result by commutative algebra completes the proof).

□

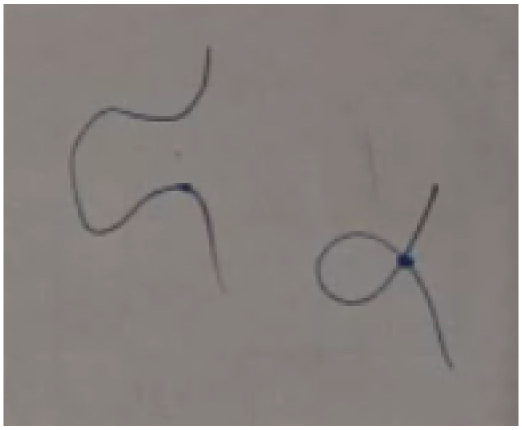


FIGURE 7. The left curve has a one parameter family around a point, whereas the right curve has a two parameter family.

LECTURE 13

9. DIVISORS

More details: Hartshorne Chapter II.6

Moral: codimenison 1 subscheme are the easiest closed subschemes to study, because they correspond to height 1 ideals, in good cases they are principal.

Recall: If $Z \subset X$ is closed and irreducible, then the codimension of Z in X is

$$\sup\{n : Z = Z_0 \subsetneqq \cdots \subsetneqq Z_n \subset X \text{ where the } Z_i \text{ are closed irreducible subsets}\}$$

Hypersurfaces have codimension 1.

9.1. **Weil divisors.**

Let X be a Noetherian, separated, integral scheme such that all the $\mathcal{O}_{X,x}$ of dimension one are dvr's (regular in codimension one, e.g., X smooth or normal).

Definition 9.1. A *prime divisor on* X is a closed integral subscheme of codimension one. A *Weil divisor* is an element of

$$\mathrm{Div}(X) := \bigoplus_{\text{prime divisors } Z \subset X} \underline{\mathbb{Z}}[Z].$$

We say a divisor is *effective*, if all its coefficients are non-negative.

Construction: the divisor of a rational function. Let $f \in \kappa(X) = \mathcal{O}_{X,\eta}$, then $\mathrm{div}(f) := \sum_{\text{prime divisors } Y \subset X} \mathrm{ord}_Y(f)[Y]$ where $\mathrm{ord}_Y(f)$ is the valuation of f in $\mathcal{O}_{X,\eta_Y}$.

Heuristically think of these as 'sums of zeros minus poles with multiplicities'

Proposition 9.2. $\mathrm{div}(f)$ *is a divisor, i.e., the sum is finite*

Proof. Use the fact that X is quasicompact. □

Definition 9.3. A *principal divisor* is $\mathrm{div}(f)$ for some $f \in \kappa(X)$.

The principal divisors form a subgroup since $\mathrm{div}(f) + \mathrm{div}(g) = \mathrm{div}(fg)$.

Definition 9.4. The *class group* of X is $\mathrm{Cl}(X) := \mathrm{Div}(X)/\{\text{principal divisors}\}$

9.2. **Calculations.**

(1) Let $X = \operatorname{Spec} A$ with A a ufd, then $\mathrm{Cl}(X) = 0$, i.e., every prime divisor will be principal.
(2) Let $X = \mathbb{P}^n_k$, then $\mathrm{Cl}(X) \simeq \mathbb{Z}$ is generated by $[H]$ where $H := \{x_0 = 0\}$ (this is true if we replace k with $\mathbb{Z}$).

We prove the second fact.

Proof. Define the degree map

$$\deg : \mathrm{Div}(\mathbb{P}^n_k) \to \mathbb{Z};$$
$$\sum n_Y[Y] \mapsto \sum n_y \deg(Y)$$

where $\deg(Y)$ is the degree of hypersurface Y. Let's extend $\mathrm{div}(-)$ to all functions on $\mathbb{P}^n_k$: Let $g \in k[x_0, \dots, x_n]$ be homogeneous of degree d, then $g = g_1^{n_1} \cdots g_r^{n_r}$ where the g_i are irreducible of degree d_i, so g_i defines a hypersurface Y_i of degree d_i, thus define $\mathrm{div}(g) := \sum n_i[Y_i] \in \mathrm{Div}(\mathbb{P}^n_k)$. $\kappa(\mathbb{P}^n_k)$ consists of g/h; where g, h are homogeneous of the same degree, so $\mathrm{div}(g/h) = \mathrm{div}(g) - \mathrm{div}(h)$ has degree zero, hence deg is a surjective group homomorphism with $d[H] \mapsto d$ for all $d \in \mathbb{Z}$. Now let $d := \deg \underline{D}$ for some $D \in \mathrm{Div}(\mathbb{P}^n_k)$, and write $D = D_1 - D_2$ with D_1, D_2 effective of degree d_1, d_2. Let $D_i := \mathrm{div}(g_i)$ for some homogeneous g_i because of the bijection

$$\{\text{Irreducible hypersurfaces in } \mathbb{P}^n_k\}$$
$$\leftrightarrow \{\text{Homogeneous prime ideals of height one in } k[x_0, \dots, x_n]\}.$$

Taking powers and products implies such an ideal is principal, and we get any D_i as $\mathrm{div}(g_i)$. Now $D - dH = \mathrm{div}(f)$ where $f = g_1/g_2x_0^d \in \kappa(\mathbb{P}^n_k)$, hence $D \sim d[H]$ in $\mathrm{Cl}(\mathbb{P}^n_k)$. ☐

Proposition 9.5. (1) *If $Z \not\subset X$ is a closed subscheme, and $U = X \setminus Z$ is the open complement, then* $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ *given by intersection, is surjective.*
(2) *If* $\operatorname{codim} Z \geq 2$, *then the previous morphism is an isomorphism.*
(3) *If* $\operatorname{codim} Z = 1$, *and Z is irreducible, then we get an exact sequence*

$$\mathbb{Z} \xrightarrow{1 \mapsto [Z]} \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0$$

called an excision sequence.

Corollary 9.6. *Let $U := \mathbb{P}^n_k \setminus$ degree d hypersurface, then* $\mathrm{Cl}(U) \simeq \mathbb{Z}/d\mathbb{Z}$.

$$X \simeq X' \implies \mathrm{Cl}(X) \simeq \mathrm{Cl}(X')$$
$$\mathrm{Cl}(X \times \mathbb{A}^1) \simeq \mathrm{Cl}(X).$$

9.3. **Cartier divisors.** Let X be Noetherian, separated, integral scheme. Recall: if D is principal, then $D = \mathrm{div}(f)$ for some $f \in \kappa(X)^\times = K^\times$ defined up to $\mathcal{O}_X^\times(X) \subset K^\times$, so D gives a section of $K^\times/\mathcal{O}^\times$.

Definition 9.7. A *Cartier divisor* on X is a global section of the sheaf $K^\times/\mathcal{O}^\times$. It is given by $X = \bigcup_{i \in i} U_i$, with $f_i \in K^\times$ such that

$$\frac{f_i}{f_j}|_{U_i \cap U_j} \in \mathcal{O}^\times(U_i \cap U_j)$$

and we identify Cartier divisors given by refining the open cover and also $(U_i, f_i) \sim (U_i, \beta_i f_i)$ for $\beta_i \in \mathcal{O}^\times(U_i)$. They form a group Cartier(X) via multiplication of the f's. A Cartier divisor is *principal*, if it is given by a rational function $f \in K^\times$: $(U_i, f\beta_i)$ where $\beta_i \in \mathcal{O}^\times(U_i)$.

$$\mathrm{CaCl}(X) := \mathrm{Cartier}(X)/\{\text{principal divisors}\}$$

9.4. **Cartier to Weil.** Assume X is a integral, Noetherian, separated scheme regular in codimension 1. Fix $D = (U_i, f_i)$. For all $Y \subset X$ codimension 1 integral subscheme, then there exists an i such that $\eta_Y \in U_i$ and we define $n_Y := \mathrm{val}_{\mathcal{O}_{\eta_Y}}(f_i)$, (note that this last quantity does not change under $f_i \mapsto \beta_i f_i$ whenever $\beta_i \in \mathcal{O}^\times(U_i)$). Define $D \in \mathrm{Cart}(X) \mapsto \sum n_Y[Y] \in \mathrm{Div}(X)$.

Theorem 9.8. *With X as above, all local rings are ufd's (e.g. X smooth over k) and* $\mathrm{Cart}(X) \xrightarrow{\sim} \mathrm{Div}(X)$*, and this correspondence sends principal Cartier divisors to principal Weil divisors exactly, so* $\mathrm{CaCl}(X) \xrightarrow{\sim} \mathrm{Cl}(X)$*.*

Moral:

- Cartier divisors are Weil divisors that are 'locally principal'.
- Local rings are ufd's, so every prime divisor is locally principal.

Example 9.9 (Non-example)**.** If X is singular, then the isomorphism can fail! Let $X = \mathrm{Spec}\, k[x,y,z]/(xy - z^2) \subset \mathbb{A}^3_k$. Now $\mathrm{CaCl}(X) = 0$ but $\mathrm{Cl}(X) = \mathbb{Z}/2$ generated by $\{y = z = 0\}$. At $\{0\} \in Z$ one needs two equations to cut out Z, one equation is not enough for any open contain U containing 0, so we have a non-locally-principal Weil divisor!

Lecture 14

Last time:

- 'Weil divisors' $\mathrm{Cl}(X) = \mathrm{Div}(X)/\text{principal}$ where $\mathrm{Div}(X)$ is the set of linear combinations of closed integral subschemes of codimension 1.
- 'Cartier divisors' $\mathrm{CaCl}(X) = \{(U_i, f_i) : f_i \in K(X)^\times, f_i/f_j \in \mathcal{O}^\times(U_i \cap U_j)\}/\text{principal}$. Think of these is locally-principal Weil divisors.

Definition 9.10. The *Picard group* of a scheme X is the group

$$\mathrm{Pic}(X) := (\text{line bundles on } X \text{ up to isomorphism}, \otimes),$$

the inverse is given by $\mathscr{L}^{-1} = \mathscr{L}^\vee$.

There is a canonical map

$$\begin{aligned} \mathrm{CaCl}(X) &\to \mathrm{Pic}(X); \\ D := (U_i, f_i) &\mapsto \mathcal{O}(D) \subset K \\ U_i &\mapsto f_i^{-1}\mathcal{O}(U_i) \qquad\qquad \alpha_{ij} = \frac{f_i}{f_j} \in \mathcal{O}^\times(U_{ij}) \\ \text{principal} &\mapsto \text{trivial line bundle} \\ (U_i, f) &\mapsto f^{-1}\mathcal{O} \sim \mathcal{O} \end{aligned}$$

Claim: $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$ is an isomorphism for X integral, Noetherian, seprarable.

Proof. We will prove this later using the cohomology of X, although one could check by hand. ☐

Example 9.11 (Check this for $n = 1$)**.**

$$\begin{aligned} \overbrace{\mathrm{Cl}(\mathbb{P}^2)}^{\text{subschemes}} \ \simeq\ \overbrace{\mathrm{CaCl}(\mathbb{P}^n)}^{\text{sheaf sections}} \ \simeq\ \overbrace{\mathrm{Pic}(\mathbb{P}^n)}^{\text{line bundles}} \\ H = \{x_\bullet = 0\} \leftrightarrow (U_i \simeq \mathbb{A}^n, f_i = x_0/x_i) \leftrightarrow \mathcal{O}(1) \\ mH \leftrightarrow (U_i \simeq \mathbb{A}^n, f_i = (x_0/x_i)^n) \leftrightarrow \mathcal{O}(m) \end{aligned}$$

10. Čech cohomology

Goal:

$$\begin{aligned} &\text{singular cohomology of topological spaces with coefficients of abelian groups} \\ &\quad \rightsquigarrow \text{cohomology of schemes with coefficients in sheaves of abelian groups.} \end{aligned}$$

This gives interesting and computable invariants on the RHS.

10.1. Definition and examples.

Let X be a topological space, and $\mathscr{F}$ a sheaf of abelian groups on X. Let $\{U_i\}_{i\in I}$ be an open cover of X with I fully ordered. Define $U_{i_0\cdots i_p} := U_{i_0} \cap \cdots \cap U_{i_p}$.

Definition 10.1. The group of (Čech) p-cohcains is

$$C^p_U(X; \mathscr{F}) := \prod_{i_0 < \cdots < i_p} \mathscr{F}(U_{i_0\cdots i_p}) \quad p \geq 0.$$

FIGURE 8. An open cover $\{U, V\}$ of S^1.

The differential is $d^p : C^p_U \to C^{p+1}_U$ is given by

$$(d\alpha)_{i_0 \cdots i_p} := \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \cdots \widehat{i_k} \cdots i_{p+1}}|_{U_{i_0 \cdots i_{p+1}}}$$

for each $\alpha \in C^p_U$.

Example 10.2. Consider

$$\begin{aligned} d^0 : \overbrace{\prod \mathscr{F}(U_i)}^{C^0} &\to \overbrace{\prod \mathscr{F}(U_{ij})}^{C_1} \\ (s_i) &\mapsto (s_j|_{U_i \cap U_j} - s_i|_{U_i \cap U_j}). \end{aligned}$$

Now consider

$$\begin{aligned} d^1 : \overbrace{\prod_{i<j} \mathscr{F}(U_{ij})}^{C^1} &\to \overbrace{\prod_{i<j<k} \mathscr{F}(U_{ijk})}^{C^2} \\ (s_{ij}) &\mapsto (s_{jk}|_{U_{ijk}} - s_{ik}|_{U_{ijk}} + s_{ij}|_{U_{ijk}}). \end{aligned}$$

It is easy to check that $d^2 = 0$, so that $C^*(X; \mathscr{F})$ is a *chain complex*

Definition 10.3. The *Čech cohomology groups of* X are:

$$H^p_U(X; \mathscr{F}) := \frac{\ker(d^p : C^p_U \to C^{p+1}_U)}{\operatorname{im}(d^{p-1} : C^{p-1}_U \to C^p_U)}$$

. Observations

1. $H^0_U(X; \mathscr{F}) = \Gamma(X; \mathscr{F}) = \mathscr{F}(X)$ because $\mathscr{F}$ is a sheaf (note $H^0 = \ker d^0$).
2. $H^m_U(X; \mathscr{F})$ for $m \geq |I|$ if I is finite. By construction there is not such $U_{i_0 \cdots i_p}$ for $p \geq |I|$.
3. (Fact) $H^*_U(X; \mathscr{F})$ does *not* depend on the ordering of U.

Remark 10.4. If one picks a 'bad' open cover U, then one gets 'bad' cohomology H^*, e.g. $U = \{X\}$ only detects $H^0_U = \mathscr{F}(X)$, so no new invariants!

Example 10.5. Let $X = S^1$, $\mathscr{F} = \underline{\mathbb{Z}}$, with open cover $\{U, V\}$ as in the above figure. Then $C^0 = C^1 = \mathbb{Z}^2$ and

$$\begin{aligned} d : C^0 &\to C^1 \\ (a, b) &\mapsto (b - a, b - a) \end{aligned}$$

so $H^0 = H^1 = \mathbb{Z}$, just like singular cohomology!

Exercise 10.6. Let $\mathscr{F} = \mathcal{O}_{\mathbb{P}^1}(-2)$, with $U = \mathbb{A}^1 \cup \mathbb{A}^1$ a cover of $\mathbb{P}^1$. Then $H^0 = 0$ but $H^1 = k$ (notice we have more information than just H^0). Help to compute:

$$C^0_U(X, \mathcal{O}(-2)) = k\left[\frac{x_1}{x_0}\right] \times k\left[\frac{x_0}{x_1}\right]$$

$$C^1_U(X, \mathcal{O}(-2)) = k\left[\frac{x_1}{x_0}\right]_{\frac{x_1}{x_0}} = k\left[\frac{x_1}{x_0}, \frac{x_0}{x_1}\right]$$

$$d(f,g) = g - f\frac{x_1^2}{x_0^2}$$

Theorem 10.7 (Homological algebra)**.** *Let X be a separated quasicompact scheme. Let $\mathscr{F} \in$ **QCoh**(X). Then $H^*_U(X; \mathscr{F})$ is independent of the choice of* finite affine open cover U*. Thus, we can denote the cohomology as just $H^*(X; \mathscr{F})$.*

Remark 10.8. Such X and $\mathscr{F}$ are good enough for us (more generally one has to take a limit of cohomology groups along such U).

Cool fact (non-examinable): Let X be a topological space, and take $\underline{A}$ to be a constant sheaf on X. If X is homotopy equivalent to a CW-complex (e.g. a K manifold), then

$$\underbrace{H^*(X, \underline{A})}_{\text{Čech cohomology}} \simeq \underbrace{H^*(X; A)}_{\text{singular cohomology}} . \tag{10.1}$$

10.2. **Cohomology of affine schemes.**

Theorem 10.9. *Let $X = \operatorname{Spec} R$ be an affine scheme and let $\mathscr{F} \in$ **QCoh**(X) and let $U = \{U_i\}$ be a finite open cover of X, then $H^n_U(X; \mathscr{F}) = 0$ for $n \geq 1$.*

Intuition:

$$\text{schemes} \leftrightsquigarrow \text{manifolds} \tag{10.2}$$

$$\text{affine schemes} \leftrightsquigarrow \mathbb{C}^n\text{'s: } H^*(\mathbb{C}^n) = 0 \text{ for } * \geq 1 \tag{10.3}$$

Proof. Next time. ☐

10.2.1. *How to show that $H^* = 0$.*

Definition 10.10. Let C^* be a chain complex $\{C^i\}_{i\in\mathbb{Z}}$ with boundary maps $d_i : C^i \to C^{i+1}$ (s.t. $d^2 = 0$). We say $f = \{f^n : C^n \to C^n\}_n$ is a *chain map* if $f \circ d = d \circ f$. Such an f induces $f : H^n \to H^n$ for each n via $[c] = [fc]$. A *chain homotopy* between chain maps f and g is a map $h = \{h^n : C^n \to C^{n-1}\}_n$ such that $f - g = d \circ h + h \circ d$. If h exists, then $f = g : H^n \to H^n$ because for any $c \in H^n$, $dc = 0$ implies $[fc - gc] = [d^c hc] = 0$.

A trick to show that $H^*(C^*) = 0$ is to show there is a homotopy between id and 0.

Lecture 15

We begin by promising the following theorem:

Theorem 10.11. *Let X be an affine scheme, $\mathscr{F} \in \mathbf{QCoh}(X)$. Then $H^m(X;\mathscr{F}) = 0$.*

Theorem 10.12 (That uses the promised theorem in the proof?)**.** *Let X be a separated quasicompact scheme, and let $\mathscr{F} \in \mathbf{QCoh}(X)$, then $H^*_U(X;\mathscr{F})$ is independent of the choice of finite open cover U.*

We now give a proof of the promised theorem.

Proof. Write $X = \operatorname{Spec} A$. Assume $U = \bigcup_{i=1}^n D(f_i)$ with $f_i \in A$. Since $\mathscr{F}$ is a quasicoherent sheaf on $\operatorname{Spec} A$, it follows $\mathscr{F} = \widetilde{M}$ where M is an A-module. We need to show

$$0 \to M \to \prod_{i_0} M_{f_{i_0}} \to \prod_{i_0<i_1} M_{f_{i_0} f_{i_1}} \to \cdots$$

is exact. It suffices to show that this sequence is exact after $(-)_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$ (look at all the stalks). Fix $\mathfrak{p}$. Choose i_{fix} such that $f_{i_{\text{fix}}} \notin \mathfrak{p}$ so that $f_{i_{\text{fix}}}$ acts faithfully on $M_\mathfrak{p}$. Define homotopy

$$h : \prod M_{f_{i_0}\cdots f_{i_{p+1}},\mathfrak{p}} \to \prod M_{f_{i_0}\cdots f_{i_p},\mathfrak{p}}$$

via the projection map

$$h(s)_{i_0\cdots i_p} = s_{i_{f_{\text{fix}}} i_0 \cdots i_p}.$$

Then $(dh + hd)(s) = s = (id - 0)(s)$, hence h gives the homotopy to show that the sequence is acyclic by the trick. For a general U refine U to distinguished open sets (skip)[2]. □

Corollary 10.13. *By a similar method, one can show that if X is a (compact) irreducible scheme, and $\underline{A}$ is a constant sheaf on X, then $H^m_X(X, \underline{A}) = 0$ for all $m > 0$.*

Remark 10.14. In general: $H^*(X;\mathscr{F}) := \operatorname{colim}_U H^*_{\underline{U}}(X;\mathscr{F})$. There exists a map $\underline{U} \to \underline{V}$ if it is a refinement, that is for all j there is an i such thaT $V_j \subset U_i$.

Remark 10.15. There is a different notion: One can define sheaf cohomolgy via derived functors. Let X be a separated Noetherian scheme and let $\mathscr{F} \in \mathbf{QCoh}(X)$, then the sheaf cohomology is the same as the Čech cohomology.

10.3. A long exact sequence on H^*.

Lemma 10.16. *Let $U \subset X$ be an open affine subscheme and let*

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$$

be a short exact sequence in $\mathbf{QCoh}(X)$, then

$$0 \to \mathscr{F}_1(U) \to \mathscr{F}_2(U) \to \mathscr{F}_3(U) \to 0$$

is exact.

---

[2] See the Stacks Project with Tag 01 × 8.

Proof. It is enough to check this on stalks. We can assume $\mathscr{F}_i|_U = \widetilde{M_i}$ and

$$0 \to \widetilde{M_1} \to \widetilde{M_2} \to \widetilde{M_3} \to 0$$

is exact if and only if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is exact (again because of stalks). □

Remark 10.17. Let X be a non-affine scheme, then $\Gamma(X, -)$ is only left exact in general.

Theorem 10.18. *Let X be a separated quasicompact scheme and let*

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$$

be a short exact sequence in $\mathbf{QCoh}(X)$. *Then there exist a long exact sequence*

$$0 \to H^0(X; \mathscr{F}_1) \to H^0(X; \mathscr{F}_2) \to H^0(X; \mathscr{F}_3) \xrightarrow{\delta} H^1(X; \mathscr{F}_1) \to \cdots$$

Proof. Take U to be an affine open cover. It is a fact that if X is a separated scheme, then any $U_{i_0 \cdots i_p}$ is also affine. By the above lemma we have for each I that

$$0 \to \mathscr{F}_1(U_I) \to \mathscr{F}_2(U_I) \to \mathscr{F}_3(U_I) \to 0$$

is a short exact sequence, and hence so is

$$0 \to C^*_U(\mathscr{F}_1) \to C^*_U(\mathscr{F}_2) \to C^*_U(\mathscr{F}_3) \to 0.$$

The claim then follows by homological algebra. □

10.3.1. *Product on the Čech cohomology.* Let $(X, \mathcal{O}_X)$ be a ringed space, then there exists a map

$$H^P_U(X; \mathscr{F}) \times H^P_U(X, \mathscr{G}) \to H^{p+q}(X; \mathscr{F} \otimes_{\mathcal{O}_K} \mathscr{G})$$
$$[(s_I), (t_I)] \mapsto (s_I \otimes t_I)$$

Remark 10.19. If $\mathscr{F} = \mathscr{G} = \underline{\mathbb{Z}}$, then $\underline{\mathbb{Z}} \otimes_{\mathcal{O}_K} \underline{\mathbb{Z}} \simeq \underline{\mathbb{Z}}$, and for X homotopic to a CW complex, this recovers X on H^*_{sing}.

10.4. **Cohomology of $\mathbb{P}^r$.**

Theorem 10.20. *Consider $\mathbb{P}^r_k$ with structure sheaf $\mathcal{O}(d)$ with $r \geq 1$ and $d \in \mathbb{Z}$. Then*

1. $H^0(\mathbb{P}^r_k, \mathcal{O}(d)) \simeq k[x_0, \dots x_r]_d$
2. $H^i(\mathbb{P}^r_k, \mathcal{O}(d)) = 0$ *for* $0 < i < r$
3. $H^r(\mathbb{P}^r_k, \mathcal{O}(-r-1)) \simeq k$
4. *The canonical map*

$$H^0(\mathbb{P}^r, \mathcal{O}(d)) \times H^r(\mathbb{P}^r_k, \mathcal{O}(-d-r-1)) \xrightarrow{\text{mult.}} H^r(\mathbb{P}^r_k, \mathcal{O}(-r-1)) \sim k$$

is a perfect pairing, i.e. the LHS consist of the Cartesian product of dual vector spaces.

Remark 10.21. The same is true for all $\mathbb{P}^n_R$.

Remark 10.22. $H^i(\mathbb{P}^r, \mathcal{O}(d)) = 0$ for $i > r$ because $\mathbb{P}^r = \bigcup_{i=1}^{n+1}$ affine open sets.

Proof. Consider $\mathscr{F} = \bigoplus_{d\in\mathbb{Z}} \mathcal{O}(d)$, a quasicoherent sheaf on $\mathbb{P}^r_k$. It is enough to compute $H^*(\mathscr{F})$ (because H^* commutes with $\bigoplus$ on Noetherian schemes). Let $S = k[x_0, \dots, x_r]$ with standard affine cover $U_i := \{x_i \neq 0\}$. We claim that $\mathscr{F}(U_{i_0\cdots i_p}) \simeq S_{x_{i_0}\cdots x_{i_p}}$, and that is an isomorphism of graded rings, where $\deg(x_{j_1}^{\ell_1} \cdots x_{j_m}^{\ell_m} = \ell_1 + \cdots + \ell_m$.

$$(\mathcal{O}(d)\text{-sections} \to \text{mononomials of degree } d).$$

Then $C^\bullet(U, \mathscr{F})$:

$$\prod S_{x_{i_0}} \to \prod S_{x_{i_0}x_1} \to \cdots \to S_{x_{i_0}\cdots x_{i_r}}.$$

(1) $H^0 = \ker d_0 \simeq S$ and respects the grading.

(3) $H^r = \operatorname{coker} d^{r-1} = \operatorname{coker}(\prod S_{x_0\cdots\widehat{x_k}\cdots x_r} \to S_{x_0\cdots x_r})$. Compute H^r:

$$\begin{aligned} S_{x_0\cdots x_r} &= \bigoplus k\{x_0^{\ell_0}\cdots x_r^{\ell_r} : \ell_i \in \mathbb{Z}\} \\ &\supset \bigoplus k\{x_0^{\ell_0}\cdots x_r^{\ell_r} : \ell_i \geq 0\} \\ &= \operatorname{im} d^{r-1}. \end{aligned}$$

Hence, $H^r(\mathbb{P}^r; \mathscr{F}) = \bigoplus k\{x_0^{\ell_0}\cdots x_r^{\ell_r} : \ell_i < 0\}$. In particular, in degree $-r-1$ the only such mononomial is $1/(x_0\cdots x_r)$.

(4) If $d < 0$, then $H^0(\mathbb{P}^r, \mathcal{O}(d)) = 0$ and $H^r(\mathbb{P}^r, \mathcal{O}(-d-r-1)) = 0$ because $-d-r-1 > -r-1$ and there are no 'negative' mononomials of such degree. If $d \geq 0$, then $H^0(\mathbb{P}^r, \mathcal{O}(d)) = \bigoplus k\{x_0^{\ell_0}\cdots x_r^{\ell_r} : \ell_i \geq 0, \{\ell_i = d\}\}$.

□

Lecture 16

11. Cohomology, divisors and miracles

11.1. Pic, CaCl, and H^1 went to a party.

Definition 11.1. Call $\mathcal{O}_X^\times \subset \mathcal{O}_X$ be the *sheaf of invertible functions*:

$$\mathcal{O}_X^*(U) := \{f \in \mathcal{O}_X(U) : \text{ there exists a } g \in \mathcal{O}_X(U) \text{ such that } fg = 1\}.$$

This is a sheaf of abelian groups under multiplication.

Theorem 11.2. $\underbrace{\mathrm{Pic}(X)}_{\textit{line bundles up to } \simeq} \simeq H^1(X, \mathcal{O}_X^*)$ *as groups.*

Proof. We construct a bijection. We want to show:

$$\left\{ \text{iso. classes of line bundles that admit a trivialisation on an open cover } \bigcup_i U_i \right\} \leftrightarrow H^1_{U_i}(X, \mathcal{O}_X^*)$$

and then take $\mathrm{colim}_{\{U_i\}}$ on both sides. Fix $X = \bigcup_i U_i$. Take a line bundle $\mathscr{L}$. it is encoded by isomorphisms of $\mathcal{O}_{U_{ij}}$-modules $\alpha_{ij} : \mathcal{O}_{U_{ij}} \xrightarrow{\sim} \mathcal{O}_{U_{ij}}$. Each α_{ij} is multiplication by an element in $\mathcal{O}^*(U_{ij})$. We have cocycle conditions $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ on U_{ijk}. Rewrite the cocycle conditions in the form $\alpha_{ij} \circ \alpha_{ik}^{-1} \circ \alpha_{jk} = 1$, which is the multiplicative form of $s_{ij} - s_{ik} + s_{jk} = 0$. We thus get

- $(\alpha_{ij}) \in H^1_{U_i}(X, \mathcal{O}_X^*)$
- $\mathscr{L} \otimes \mathscr{L}'$ corresponds to $(\alpha_{ij} \circ \alpha'_{ij})$

Claim: $[(\alpha_{ij})] = [(\widetilde{\alpha_{ij}})]$ are in $H^1_{\{U_i\}}(X, \mathcal{O}_X^*)$ if (α_{ij}) and $(\widetilde{\alpha_{ij}})$ give isomorphic line bundles. In H^1: $[(\alpha_{ij})] = [(\widetilde{\alpha}_{ij})]$ if and only if $\alpha_{ij} = \beta_j \circ \widetilde{\alpha_{ij}} \circ \beta_i^{-1}$ for $\beta_i \in \mathcal{O}_{U_i}^*$, $\beta_j \in \mathcal{O}_{U_j}^*$ (in additive notation: $(s_i) \in C^\infty \rightsquigarrow d(s_i) = s_j - s_i$ on U_{ij}). In line bundles:

$$\begin{tikzcd}
\mathcal{O}_{U_{ij}} \arrow[d, "\sim"', "\widetilde{\alpha_{ij}}"] \arrow[rrr, bend left, "\beta_i"] & \widetilde{\mathscr{L}}|_{U_{ij}} \arrow[l, "\sim"'] \arrow[d, equal] & \mathscr{L}|_{U_{ij}} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, equal] & \mathcal{O}_{U_{ij}} \arrow[d, "\sim"', "\alpha_{ij}"] \\
\mathcal{O}_{\mathcal{U}_{|\rangle}} \arrow[rrr, bend right, "\beta_j"'] & \widetilde{\mathscr{L}}|_{U_{ji}} \arrow[l, "\sim"'] & \mathscr{L}|_{U_{ji}} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathcal{O}_{U_{ji}}
\end{tikzcd}.$$

Taking $\mathscr{L} = \widetilde{\mathscr{L}}$ with a different trivialisation shows that $[\mathscr{L}] \in H^1$ does not change! □

Theorem 11.3. *Let X be an integral Noetherian separated scheme. Then* $\mathrm{CaCl}(X) \simeq H^1(X, \mathcal{O}_X^*)$.

Corollary 11.4. $\mathrm{CaCl}(X) \simeq \mathrm{Pic}(X); D \mapsto \mathcal{O}(D)$, *in particular, $D \sim D'$ if and only if $\mathcal{O}(D) \simeq \mathcal{O}(D')$.*

Proof of the Theorem. Consider the short exact sequence of sheaves

$$0 \to \mathcal{O}_X^* \to K^* \to K^*/\mathcal{O}_X^* \to 0$$

and then take the long exact sequence

$$0 \to H^0(X; \mathcal{O}_X^*) \to H^0(X; K^*) \to \overbrace{H^0(X; K^*/\mathcal{O}_X^*)}^{\text{Cartier divisors}}$$
$$\xrightarrow{\delta} H^1(X; \mathcal{O}_X^*) \to H^1(X; K^*) \to \cdots .$$

Note $H^1(X; K^*) = 0$ because K^* is constant and X is irreducible. By definition $\mathrm{CaCl}(X) = H^0(X; K^*/\mathcal{O}_X^*)/\operatorname{im} H^0(X; K^*)$ which is isomorphic to $H^1(X; \mathcal{O}_X^*)$ by the above long exact sequence. □

We conclude with

$$\begin{aligned} H^1(X, \mathcal{O}_X^*) &\simeq \mathrm{Pic}(X) && \text{always} \\ &\simeq \mathrm{CaCl}(X) && X \text{ is integral, Noetherian, and separated} \\ &\simeq \mathrm{Cl}(X). && X \text{ is also regular in codimension 1} \end{aligned}$$

11.1.1. *Functorality of* Cl.

Proposition 11.5. (1) *If $f : X \to Y$ is a flat morphism of schemes, then $f^* \operatorname{Div}(Y); z \mapsto f^{-1}(z)$, so that each $f^{-1}(z)$ is of codimension 1 because of flatness. This map factors through* Cl

(2) *If $f : X \to Y$ is a proper morphism of schemes, then*

$$f_* : \operatorname{Div}(X) \to \operatorname{Div}(Y)$$
$$z \mapsto \begin{cases} \overline{f(z)} & \textit{if it is a prime divisor} \\ 0 & \textit{otherwise.} \end{cases}$$

Note that each $\overline{f(z)}$ is closed and irreducible but not necessarily of codimension 1. This map factors through Cl *when f is proper.*

11.2. **Satz von Riemann–Roch.** Recall:

$$D \text{ a Cartier divisor on } X \rightsquigarrow \mathcal{O}_X(D) \text{ line bundle}$$
$$(U_i, f_i) \mapsto \frac{1}{f_i}\mathcal{O}_X(U_i) \text{ on } U_i.$$

More generally:

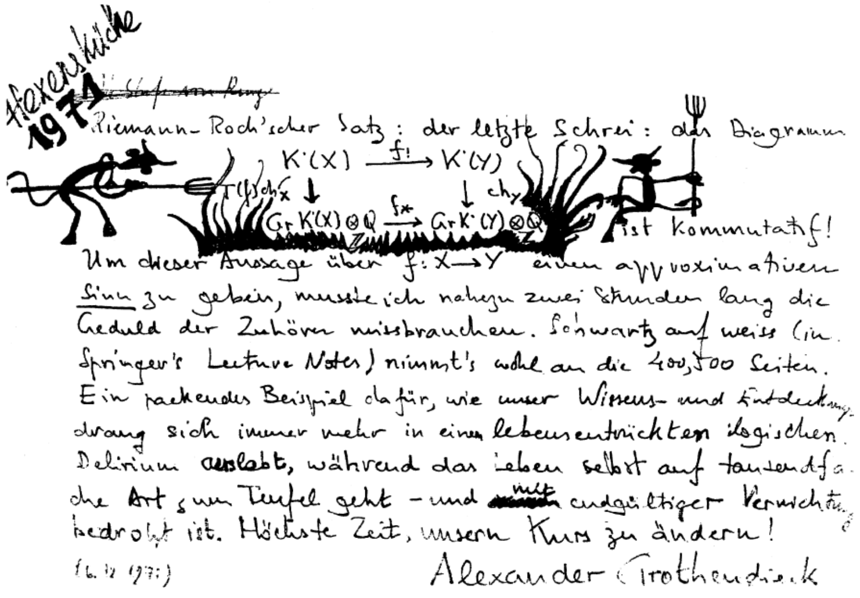
$$D \text{ a Weil divisor} \rightsquigarrow \mathcal{O}_X(D) \text{ an } \mathcal{O}_X\text{-module}$$
$$\mathcal{O}_X(D) : U \mapsto \{0\} \cup \{f \in K : \operatorname{div}(f) + D \geq 0\}$$

where $D \geq 0$ means that all the coefficients of D are nonnegative.

Example 11.6. Consider figure 10. An f should have a divisor of order at least 3 at $\{0\}$ allowed to have a pole at most $\{1\}$ and no more poles!

- $\mathcal{O}_X(D)$ is an $\mathcal{O}_X$-module.
- $\mathcal{O}_X(D)$ is a line bundle if and only if D is locally principal (a Cartier divisor) because: if there is an open cover $\{U_i\}$, then

$$\mathcal{O}_X(U_i) \xrightarrow{\sim} \Gamma(U_i, \mathcal{O}_X(D))$$
$$1 \mapsto f_i \in K$$



Hexenküche 1971

Riemann-Roch'scher Satz: der letzte Schrei: das Diagramm

$$\begin{tikzcd} K^\cdot(X) \arrow[r, "f_!"] \arrow[d, "T(f)ch_X"'] & K^\cdot(Y) \arrow[d, "ch_Y"] \\ Gr\, K^\cdot(X) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r, "f_*"] & Gr\, K^\cdot(Y) \otimes_{\mathbb{Z}} \mathbb{Q} \end{tikzcd}$$

ist kommutativ! Um dieser Aussage über $f : X \to Y$ einen approximativen Sinn zu geben, musste ich nahezu zwei Stunden lang die Geduld der Zuhörer missbrauchen. Schwarz auf weiss (in Springer's Lecture Notes) nimmt's wohl an die 400, 500 Seiten. Ein packendes Beispiel dafür, wie unser Wissens- und Entdeckungsdrang sich immer mehr in einem lebensentrückten logischen Delirium auslebt, während das Leben selbst auf tausendfache Art zum Teufel geht – und mit endgültiger Vernichtung bedroht ist. Höchste Zeit, unsern Kurs zu ändern!

(6.12.1971)

Alexander Grothendieck

Figure 9. A sketch of Grothedieck on Riemann–Roch.

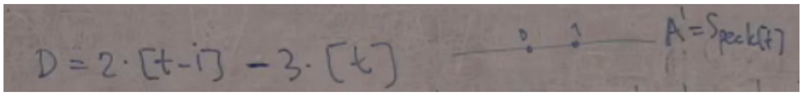


Figure 10. An example of a divisor on an affine space.

means exactly that $\widetilde{D} = (U_i, f_i)$ is a Cartier divisor and

$$\mathcal{O}_X(\widetilde{D})(U_i) =^{\text{old}} \mathcal{O}_X(U_i) =^{\text{new}} \Gamma(U_i, \mathcal{O}_X(\widetilde{D})).$$

Moreover, $\widetilde{D} \mapsto D$ under $\operatorname{Cart}(X) \to \operatorname{Div}(X)$.

The rest is non-examinable.

Theorem 11.7 (Riemann–Roch)**.** *Let C be a projective smooth algebraic curve over an algebraically closed field k. Let $D = \sum n_i[p_i]$ be a Weil divisor of degree $d = \sum n_i$. Let $\mathscr{F} := \mathcal{O}_C(D)$, define the* Euler character $\chi(C; \mathscr{F}) := \sum(-1)^m \dim_k H^m(C; \mathscr{F})$. *Then*

$$\chi(C; \mathscr{F}) = \deg D + \chi(C; \mathcal{O}_C)$$

where $\chi(C, \mathcal{O}_C) := 1 - \operatorname{genus}(C)$.

Remark 11.8. When $k = \mathbb{C}$, the quantity $\operatorname{genus}(C)$ is the same quantity as the topological genus of the Riemann surface C.

$$\text{smooth proj. alg. curves}/\mathbb{C} \hookrightarrow \text{compact Riemann surfaes}$$
$$X \mapsto X(\mathbb{C})$$

Moral: when $k = \mathbb{C}$: for a compact Riemann surface M, the number of linearly independent meromorphic functions with a chosen restriction on the poles only depends on the genus of M.

Corollary 11.9. *Let M be a compact connected Riemann surface and pick a point $a \in M$. Then there exists a non-constant function f on M which has a pole of order $\leq \operatorname{genus}(M) + 1$ at a and is holomorphic otherwise.*

Proof. Let $g := \operatorname{genus}(M)$. The divisor $D = (g+1)[a]$ has degree $g+1$, so

$$\begin{aligned}\dim H^0(M, \mathcal{O}(D)) &\geq \underbrace{\chi_m(\mathcal{O})}_{h^0 - h^1} \\ &= d - g + 1 && \text{Riemann–Roch} \\ &= g + 1 - g + 1 \\ &= 2\end{aligned}$$

and constant functions form a one-dimensional subspace, hence there exists a non-constant $f \in H^0(M, \mathcal{O}(D))$. $\square$